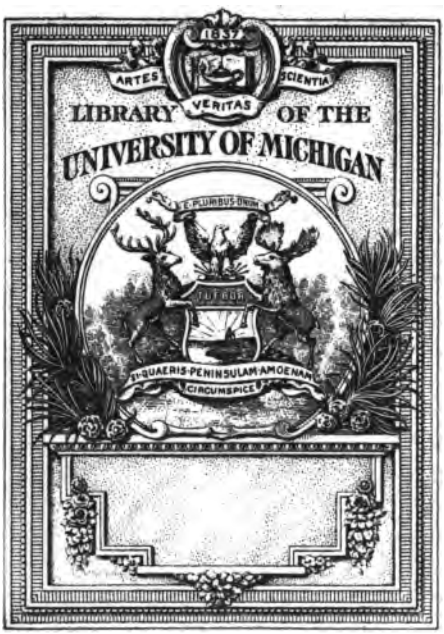
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PLANE TRIGONOMETRY.

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PLANE TRIGONOMETRY

For the Use of Colleges and Schools.

WITH NUMEROUS EXAMPLES.

BY

I. TODHUNTER, M.A., F.R.S.

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1864.

PREFACE.

THE present work contains all the propositions which are usually included in treatises on Plane Trigonometry, together with more than six hundred examples for exercise. The design has been to render the subject intelligible to beginners, and at the same time to afford the student the opportunity of obtaining all the information which he will require on this branch of Mathematics. The work is divided into a large number of chapters, each of which is in a great measure complete in itself. Thus it will be easy for teachers to select for pupils such portions as will be suitable for them in their first reading of the book. Each chapter is followed by a set of examples; those which are entitled *Miscellaneous Examples*, together with a few in some of the other sets, may be advantageously reserved by the student for exercise after he has made some progress in the subject.

As the text and the examples of the present work have been tested by considerable experience in teaching, the hope may be entertained that they will be suitable for imparting a sound and comprehensive knowledge of Plane Trigonometry, together with readiness in the application of this knowledge to the solution of problems. Any suggestions or corrections from students and teachers will be most thankfully received.

I. TODHUNTER.

ST. JOHN'S COLLEGE,
Feb. 21, 1859.

In the second edition the work has been revised, and the hints for the solution of the examples have been considerably increased.

December, 1860.

CONTENTS.

PLANE TRIGONOMETRY.

I. MEASUREMENT OF ANGLES BY DEGREES OR GRADES.

1. THE word Trigonometry is derived from two Greek words, one signifying a *triangle* and the other signifying *I measure*, and originally denoted the science in which the relations subsisting between the sides and angles of a triangle were investigated; the science was called *plane* trigonometry, or *spherical* trigonometry, according as the triangle was formed on a *plane* surface or on a *spherical* surface. Plane Trigonometry has now a wider meaning, and comprises all algebraical investigations with respect to plane angles, whether forming a triangle or not.

2. We have first to explain how angles are measured. A plane rectilineal angle is defined by Euclid as the inclination of two straight lines to one another which meet together, but are not in the same straight line. And when a straight line standing on another makes the adjacent angles equal to one another, each of the angles is called a *right angle*. A right angle is divided into 90 equal parts called *degrees*, a degree is divided into 60 equal parts called *minutes*, and a minute into 60 equal parts called *seconds*. Thus any angle may be estimated by ascertaining the number of degrees it contains; if the angle does not contain an exact number of degrees, we can express it in degrees and a fraction of a degree; or the fraction of a degree may be converted into minutes and seconds.

3. Thus, for example, half a right angle contains 45 degrees; a quarter of a right angle contains $22\frac{1}{2}$ degrees, which we may write

in the decimal notation 22·5 degrees, or we may express it as 22 degrees, 30 minutes. Similarly, if a right angle be divided into 16 equal parts, each part contains $5\frac{5}{8}$ degrees, that is, 5 degrees, 37 minutes, 30 seconds.

4. Symbols are used as abbreviations of the words *degrees*, *minutes*, *seconds*. Thus $5^{\circ}\ 37'\ 30''$ is used to denote 5 degrees, 37 minutes, 30 seconds.

5. The method of estimating angles by degrees, minutes, and seconds, is almost universally adopted in practical calculations. Another method was proposed in France in connexion with a uniform system of decimal tables of weights and measures. In this method a right angle is divided into 100 equal parts called *grades*, a grade is divided into 100 equal parts called *minutes*, and a minute is divided into 100 equal parts called *seconds*. On account of the occurrence of the number *one hundred* in forming the subdivisions of a right angle, this method of estimating angles is called the *centesimal* method; and the common method is called the *sexagesimal* method on account of the occurrence of the number *sixty* in forming the subdivisions of a degree. The eentesimal method is also called the *French* method, and the common method is called the *English* method.

6. Symbols are used as abbreviations of the words *grades*, *minutes*, and *seconds*, in the centesimal method. Thus $5^{g}\ 37^{\backprime}\ 30^{\backprime\backprime}$ is used to denote 5 grades, 37 minutes, 30 seconds in the centesimal method. A centesimal minute and second are not the same as a sexagesimal minute and second, and the accents which are used to denote centesimal minutes and seconds differ from those which are used to denote sexagesimal minutes and seconds.

7. In the centesimal method any whole number of minutes and seconds may be expressed immediately as a decimal fraction of a grade. Thus 37 minutes is $\frac{37}{100}$ of a grade, that is ·37 of a grade; and 30 seconds is $\frac{30}{(100)^2}$ of a grade, that is ·003 of a grade.

Hence $5^g\ 37'\ 30''$ may be written $5^g{\cdot}373$; and since a grade is $\left(\frac{1}{100}\right)^{\text{th}}$ of a right angle, $5^g{\cdot}373$ may be written as $\cdot 05373$ of a right angle. Notwithstanding this great advantage of the centesimal method, the sexagesimal method has been retained in practical calculations, because the latter had become thoroughly established by long use in mathematical works, and especially in mathematical tables, before the former was proposed; and such works and tables would have been rendered almost useless by the change in the method of estimating angles.

8. We will now shew how to compare the numbers which measure the same angle in the English and French methods.

Let D be the number of *degrees* contained in any angle, G the number of *grades* contained in the same angle. Then since there are 90 degrees in a right angle, $\frac{D}{90}$ expresses the ratio of the given angle to a right angle; and since there are 100 grades in a right angle, $\frac{G}{100}$ also expresses the ratio of the given angle to a right angle.

Hence $$\frac{D}{90} = \frac{G}{100};$$

therefore $$D = \frac{90}{100} G = \frac{9}{10} G = G - \frac{1}{10} G,$$

and $$G = \frac{100}{90} D = \frac{10}{9} D = D + \frac{1}{9} D.$$

The formula $D = G - \frac{1}{10} G$ gives the following rule; *From the number of grades contained in any angle subtract one-tenth of that number, the remainder is the number of degrees contained in the angle.*

The formula $G = D + \frac{1}{9} D$ gives the following rule; *To the number of degrees contained in any angle add one-ninth of that number, the sum is the number of grades contained in the angle.*

9. Again, let m be the number of English minutes contained in any angle, μ the number of French minutes contained in the same angle. Then since there are 90×60 English minutes in a right angle, $\frac{m}{90 \times 60}$ expresses the ratio of the given angle to a right angle; and since there are 100×100 French minutes in a right angle, $\frac{\mu}{100 \times 100}$ also expresses the ratio of the given angle to a right angle. Hence

$$\frac{m}{90 \times 60} = \frac{\mu}{100 \times 100};$$

therefore $$m = \frac{9 \times 6}{10 \times 10}\mu = \frac{27}{50}\mu,$$

and $$\mu = \frac{50}{27}m.$$

Similarly, if s be the number of English seconds contained in any angle, and σ the number of French seconds contained in the same angle,

$$\frac{s}{90 \times 60 \times 60} = \frac{\sigma}{100 \times 100 \times 100};$$

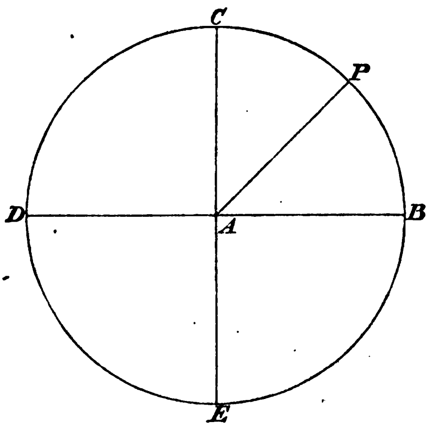
therefore $$s = \frac{81}{250}\sigma,$$

and $$\sigma = \frac{250}{81}s.$$

10. The angles considered in Geometry are *in general* less than two right angles. We say *in general*, because angles greater than two right angles are not altogether excluded. For we may refer to the proposition that in equal circles, angles, whether at the centres or the circumferences, have the same ratio which the circumferences on which they stand have to one another; here there is no limit to the magnitude of the circumferences, and consequently no limit to the magnitude of the angles; and in the course of the demonstration given by Euclid, an angle occurs which may

be ***any multiple whatever of a given angle,*** and so may be as ***great as we please.***

11. It is however usual in works on Trigonometry expressly to state that there is to be no restriction with respect to the magnitude of the angles considered. Let *BAD* be any straight line,



CAE a straight line at right angles to the former. Suppose a line *AP* to revolve round one end *A*, starting from the fixed position *AB*. When *AP* coincides in direction with *AC*, the angle which has been described is a right angle; when *AP* coincides in direction with *AD*, the angle described is two right angles; when *AP* coincides with *AE*, the angle described is three right angles; when *AP* coincides with *AB*, the angle described is four right angles. Then as *AP* proceeds through a second revolution, the angle described will be greater than four right angles. Thus if *AP* be situated midway between *AB* and *AC*, the angle between *AB* and *AP* will be *half* a right angle if *AP* be supposed in its *first* revolution; the angle will be four right angles and a half if *AP* be supposed in its *second* revolution; the angle will be eight right angles and a half if *AP* be supposed in its *third* revolution; and so on.

12. The straight lines CAE and BAD form by their intersection four right angles; these are called *quadrants*. BAC is called the *first quadrant*, CAD the *second quadrant*, DAE the *third quadrant*, and EAB the *fourth quadrant*. Now suppose any angle formed by the fixed line AB and the moveable line AP; if AP is situated in the first quadrant, the angle BAP is said to be in the first quadrant; if AP is situated in the second quadrant, the angle is said to be in the second quadrant; and so on.

EXAMPLES.

1. The difference of two angles is 10 grades and their sum is 45 degrees; find each angle.

2. Divide two-thirds of a right angle into two parts, such that the number of degrees in one part may be to the number of grades in the other part as 3 to 10.

3. Divide half a right angle into two parts, such that the number of degrees in one part may be to the number of grades in the other part as 9 to 5.

4. Find the measure of 1\` 5\`\` in decimals of a degree.

5. Divide an angle which contains n degrees into two parts, one of which contains as many English minutes as the other does French.

6. If one-third of a right angle be assumed as the unit of angular measure, what number will represent 75°?

7. Determine the number of degrees in the unit of angular measure when an angle of $66\frac{2}{3}$ grades is represented by 20.

8. The numbers of the sides of two regular polygons are as 2 to 3, and the number of grades in an angle of one equals the number of degrees in an angle of the other. Find the angles.

9. Shew that an angle expressed in centesimal seconds will be reduced to sexagesimal by multiplying by the factor ·324.

10. Compare the angles which contain the same number of English seconds as of French minutes.

II. CIRCULAR MEASURE OF AN ANGLE.

13. We have explained two methods of estimating angles, namely, that by means of *degrees* and subdivisions, and that by means of *grades* and subdivisions, and we have stated that the former method is that which is most commonly used in *practical* calculations. There is, however, another method of estimating angles which is of great importance in the *theory* of mathematics, which we shall now explain. The object of the present chapter is to establish and apply the following proposition; *If with the point of intersection of any two straight lines as centre a circle be described with any radius, then the angle contained by the straight lines may be measured by the ratio of the length of the arc of the circle intercepted between the lines to the length of the radius.* We shall require some preliminary propositions; the proposition in Art. 14 is sometimes assumed, and the beginner may adopt this course and return to the point hereafter.

14. *The circumferences of circles vary as their radii.*

Let R denote the radius and C the circumference of one circle; let r denote the radius and c the circumference of another circle. In each circle let a regular polygon of n sides be inscribed, and in each circle draw two lines from the centre to the extremities of one of the sides of the inscribed polygon; thus we obtain two *similar triangles.* Let P denote the perimeter of the polygon inscribed in the first circle, and p the perimeter of the polygon inscribed in the second circle. By similar triangles a side of the first polygon is to a side of the second polygon as the radius of the first circle is to the radius of the second circle; therefore also

$$\frac{P}{p} = \frac{R}{r}.$$

Now let $P = C - X$ and $p = c - x$; thus

$$r(C - X) = R(c - x);$$

therefore $$rC - Rc = rX - Rx.$$

Now we assume that by making n as large as we please, the perimeter of each polygon can be made to differ as *little as we please* from the circumference of the corresponding circle; thus X and x can each be made as small as we please, and therefore $rX - Rx$ can be made as small as we please. Hence $rC - Rc$ must be *zero;* for if it had any value a then $rX - Rx$ could not be made less than a, which is inconsistent with the fact that $rX - Rx$ can be made as small as we please. Thus

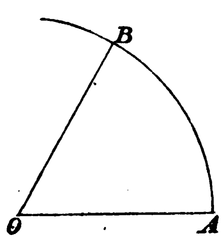
$$rC - Rc = 0,$$

therefore $$\frac{C}{R} = \frac{c}{r}.$$

15. Thus the ratio of the circumference of a circle to its radius is *constant* whatever be the magnitude of the circle; therefore of course the ratio of the *circumference* to the *diameter* is also constant. The numerical value of the ratio of the circumference of a circle to its diameter cannot be stated *exactly;* but, as we shall shew hereafter, this ratio may be calculated to any degree of approximation that is required; the value is approximately equal to $\frac{22}{7}$, and still more nearly equal to $\frac{355}{113}$; the value correct to eight places of decimals is 3·14159265... The symbol π is invariably used to denote the ratio of the circumference of a circle to its diameter; hence, if r denote the radius of a circle, its circumference is $2\pi r$, where

$$\pi = 3{\cdot}14159....$$

16. *The angle subtended at the centre of a circle by an arc which is equal in length to the radius is an invariable angle.*



With centre O and any radius OA describe a circle; let AB be an arc of this circle equal in length to the radius. Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

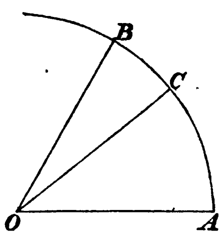
$$\frac{\text{angle } AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{circumference of the circle}} = \frac{r}{2\pi r} = \frac{1}{2\pi};$$

therefore $$\text{angle } AOB = \frac{4 \text{ right angles}}{2\pi}.$$

Thus the angle AOB is a certain fraction of four right angles which is constant, whatever may be the radius of the circle.

17. Since the angle subtended at the centre of a circle by an arc which is equal to the radius is an *invariable angle*, it may be taken as the unit of angular measurement, and then any angle will be estimated by the ratio which it bears to this unit.

Let AOC be any angle; with O as centre and any radius OA



describe a circle; let AB be an arc of this circle equal in length to the radius; let r denote the radius, and l the length of the arc AC.

Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

$$\frac{\text{angle } AOC}{\text{angle } AOB} = \frac{AC}{AB} = \frac{l}{r};$$

therefore $$\text{angle } AOC = \frac{l}{r} \times \text{angle } AOB;$$

this result is true whatever the unit of angular measurement may be, the same unit of course being used for the two angles. If we take the angle AOB itself for the unit, then this angle must be denoted by unity;

thus $$\text{angle } AOC = \frac{l}{r}.$$

18. We have thus proved that any angle may be estimated by a *fraction which has for its numerator the arc subtended by that angle at the centre of any circle, and for its denominator the radius of that circle.* And in this mode of estimating angles the unit, that is the angle denoted by 1, is the angle in which the arc subtended is equal to the radius. We have shewn that this angle is $\frac{4 \text{ right angles}}{2\pi}$; hence the number of *degrees* contained in this angle is $\frac{360}{2\pi}$, that is $\frac{180}{\pi}$. If we use the approximate value of π given in Art. 15, we shall find that $\frac{180}{\pi} = 57 \cdot 29577951 \ldots$; this therefore is the number of degrees contained in the angle which is subtended by an arc equal to the radius.

19. Thus there are two methods of forming an idea of the magnitude of an angle which is estimated by the fraction *arc divided by radius.* Suppose, for example, we speak of the angle $\frac{2}{3}$; we may refer to the *unit* of angular measurement, which is an angle containing about 57 degrees, and imagine two-thirds of this unit to be taken; or without thinking about the *unit* at all, we may suppose an angle is taken such that the arc subtending it is two-thirds of the corresponding radius.

20. The fraction *arc divided by radius* is called the *circular measure of an angle.* Since, as we have already stated, this method of measuring angles is very much used in theoretical investigations, it is sometimes called the *theoretical* method.

21. If r denote the radius of a circle, the circumference is $2\pi r$; hence the circular measure of four right angles is $\frac{2\pi r}{r}$, that is 2π.

The circular measure of two right angles is π; the circular measure of one right angle is $\frac{\pi}{2}$; and the circular measure of n right angles is $\frac{n\pi}{2}$, where n may be either integral or fractional.

22. We will now shew how to connect the *circular* measure of any angle with the measure of the same angle in *degrees*. Let x denote the number of degrees in any given angle, θ the circular measure of the same angle. Since there are 180 degrees in two right angles, $\frac{x}{180}$ expresses the ratio of the given angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the given angle to two right angles. Hence

$$\frac{x}{180}=\frac{\theta}{\pi};$$

thus

$$x=\frac{180\theta}{\pi},$$

and

$$\theta=\frac{\pi x}{180}.$$

23. For example, the circular measure of an angle of 1 degree is $\frac{\pi}{180}$; the circular measure of an angle of 10 degrees is $\frac{10\pi}{180}$; the circular measure of an angle of half a degree is $\frac{\pi}{180}\times\frac{1}{2}$; the circular measure of an angle of one minute is $\frac{\pi}{180\times 60}$; the circular measure of an angle of one second is $\frac{\pi}{180\times 60\times 60}$; and so on.

Again; if the circular measure of an angle is $\frac{3}{4}$ the number of degrees contained in the angle is $\frac{3}{4}.\frac{180}{\pi}$, that is $\frac{3}{4}$ of $57{\cdot}2957795\ldots$; if the circular measure of an angle is 10, the number of degrees contained in the angle is $10 . \frac{180}{\pi}$, that is $10 \times 57{\cdot}2957795\ldots$; and so on.

The student is recommended to pay particular attention to these points; especially he should accustom himself to express readily in circular measure an angle which is given in degrees.

24. Similarly we may connect the circular measure of any angle with the measure of the same angle in grades.

Let y denote the number of grades in any given angle, θ the circular measure of the same angle; then the ratio of the given angle to two right angles is expressed by $\frac{y}{200}$ and also by $\frac{\theta}{\pi}$. Hence

$$\frac{y}{200} = \frac{\theta}{\pi};$$

thus

$$y = \frac{200\theta}{\pi},$$

and

$$\theta = \frac{\pi y}{200}.$$

The number of grades in the angle which is the unit of circular measure is $\frac{200}{\pi}$, that is, $63{\cdot}661977\ldots$

25. In Art. 17 we proved that

$$\text{angle } AOC = \frac{l}{r} \times \text{angle } AOB;$$

where nothing is assumed respecting the unit of angular measurement, except that the *same* unit is to be employed for both angles. Since AOB is an invariable angle, we see that the magnitude of any angle AOC varies as the subtending arc *directly*, and as the radius *inversely*. Thus we may say that

$$\text{angle } AOC = \frac{k \times \text{arc}}{\text{radius}};$$

when k is some quantity which does not change with AOC, and the value of which depends upon the unit of angular measurement which we please to employ. Suppose, for example, that we wish to take the half of a right angle as our unit; then we require that AOC should be equal to 1 when the arc is the eighth part of the circumference; thus

$$1 = \frac{k \times \frac{2\pi r}{8}}{r};$$

therefore $$k = \frac{4}{\pi}.$$

Thus the formula

$$\text{angle } AOC = \frac{4}{\pi} \times \frac{\text{arc}}{\text{radius}}$$

gives the correct estimate of the magnitude of an angle when the unit is half a right angle.

EXAMPLES.

1. If D, G, C be respectively the number of degrees, grades, and units of circular measure in an angle, shew that

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}.$$

2. Find the number of degrees in the angle subtended at the centre of a circle whose radius is 10 feet by an arc whose length is 9 inches.

3. Find the circular measure of $1^g.\ 1^{\grave{}}.$

4. There are three angles; the circular measure of the first exceeds that of the second by $\frac{\pi}{10}$, the sum of the second and third is 30 grades, and the sum of the first and second is 36 degrees. Determine the three angles.

5. Express five-sixteenths of a right angle in circular measure,

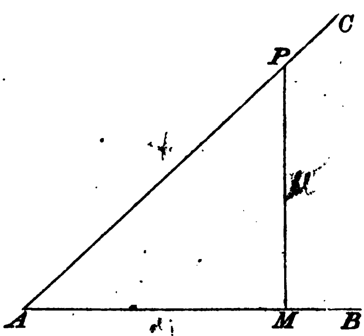
in degrees and decimals of a degree, and in grades and decimals of a grade.

6. The angles of a triangle are in arithmetical progression, and the greatest is double the least; express the angles in degrees, grades, and circular measure.

7. The angles of a triangle are in arithmetical progression, and the number of degrees in the least is to the circular measure of the greatest as 60 to π; find the angles.

III. TRIGONOMETRICAL RATIOS.

26. Let BAC be any angle; take any point in either of the containing sides, and from it draw a line perpendicular to the other



side; let P be the point in the side AC and PM perpendicular to AB. We shall use the letter A to denote the angle BAC. Then

$\frac{PM}{AP}$, that is $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is called the *sine* of the angle A;

$\frac{AM}{AP}$, that is $\frac{\text{base}}{\text{hypotenuse}}$, is called the *cosine* of the angle A;

$\frac{PM}{AM}$, that is $\frac{\textit{perpendicular}}{\textit{base}}$, is called the *tangent* of the angle A;

$\frac{AM}{PM}$, that is $\frac{\textit{base}}{\textit{perpendicular}}$, is called the *cotangent* of the angle A;

$\frac{AP}{AM}$, that is $\frac{\textit{hypotenuse}}{\textit{base}}$, is called the *secant* of the angle A;

$\frac{AP}{PM}$, that is $\frac{\textit{hypotenuse}}{\textit{perpendicular}}$, is called the *cosecant* of the angle A.

If the cosine of A be subtracted from unity, the remainder is called the *versed sine* of A. If the sine of A be subtracted from unity, the remainder is called the *coversed sine* of A; the latter term however is rarely used in practice.

27. The words *sine*, *cosine*, &c. are usually abbreviated in writing and printing; thus the above definitions may be expressed as follows,

$$\sin A = \frac{PM}{AP},$$

$$\tan A = \frac{PM}{AM},$$

$$\sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP},$$

$$\cot A = \frac{AM}{PM},$$

$$\operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A,$$

$$\operatorname{covers} A = 1 - \sin A.$$

28. The *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine* are called *trigonometrical ratios* or *trigonometrical functions*; sometimes they have been called *goniometrical*

functions. A large part of *Trigonometry* consists in the investigation of the *properties and relations of these functions of an angle.* These functions are, it will be observed, not *lengths,* but *ratios* of one length to another; that is, they are arithmetical whole numbers or fractions.

29. The defect of any angle from a right angle is called the *complement* of that angle; thus if A denote the number of degrees contained in any angle, $90 - A$ is the number of degrees contained in the *complement* of that angle. This affords another method of defining some of the Trigonometrical ratios; after defining, as in Art. 26, the *sine, tangent,* and *secant* of an angle we may say

the cosine of an angle is the sine of the complement of that angle;

the cotangent of an angle is the tangent of the complement of that angle;

the cosecant of an angle is the secant of the complement of that angle.

For in the triangle PAM the angle APM is the complement of the angle A; and

$$\sin APM = \frac{\textit{perpendicular}}{\textit{hypotenuse}} = \frac{AM}{AP} = \cos A;$$

$$\tan APM = \frac{\textit{perpendicular}}{\textit{base}} = \frac{AM}{MP} = \cot A;$$

$$\sec APM = \frac{\textit{hypotenuse}}{\textit{base}} = \frac{AP}{MP} = \operatorname{cosec} A.$$

These results may also be expressed thus:

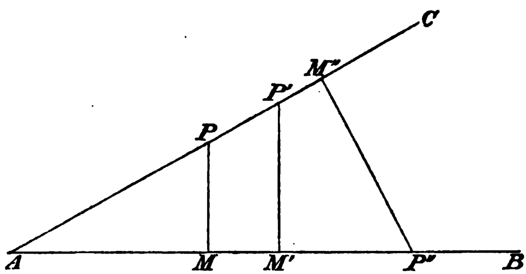
the sine of an angle is the cosine of the complement of that angle;

the tangent of an angle is the cotangent of the complement of that angle;

the secant of an angle is the cosecant of the complement of that angle.

30. ***The trigonometrical ratios remain unchanged so long as the angle remains unchanged.***

Let BAC be any angle; in AC take any point P and draw PM perpendicular to AB; also take any other point P' and draw $P'M'$ perpendicular to AB. Then by similar triangles $\frac{PM}{AP} = \frac{P'M'}{AP'}$, that is, the *sine* of the angle A is the same whether it be formed from



the triangle APM or from the triangle $AP'M'$. The same result holds for the other Trigonometrical ratios. Or we may suppose a point P'' taken in AB and $P''M''$ drawn perpendicular to AC; then the triangles APM and $AP''M''$ are similar, and $\frac{PM}{AP} = \frac{P''M''}{AP''}$.

We now proceed to establish certain relations which hold among the Trigonometrical ratios.

31. We have immediately from the definitions

$\tan A \times \cot A = 1$; therefore $\tan A = \frac{1}{\cot A}$, $\cot A = \frac{1}{\tan A}$;

$\sec A \times \cos A = 1$; therefore $\sec A = \frac{1}{\cos A}$, $\cos A = \frac{1}{\sec A}$;

$\text{cosec}\, A \times \sin A = 1$; therefore $\text{cosec}\, A = \frac{1}{\sin A}$, $\sin A = \frac{1}{\text{cosec}\, A}$.

Also

$$\tan A = \frac{PM}{AM} = \frac{PM}{AP} \div \frac{AM}{AP} = \frac{\sin A}{\cos A},$$

$$\cot A = \frac{AM}{PM} = \frac{AM}{AP} \div \frac{PM}{AP} = \frac{\cos A}{\sin A}.$$

32. *To prove that* $(\sin A)^2 + (\cos A)^2 = 1$.

In the right-angled triangle APM we have

$$PM^2 + AM^2 = AP^2;$$

therefore $$\frac{PM^2 + AM^2}{AP^2} = 1,$$

therefore $$\left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = 1;$$

that is $$(\sin A)^2 + (\cos A)^2 = 1.$$

33. With respect to the preceding proof it should be remarked that it is shewn in Euclid, I. 47, that the square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the sides; and it is known that the *geometrical* square described upon any line is measured by the *arithmetical* square of the number which measures the length of the line. From combining these two results we obtain the *arithmetical* equality

$$PM^2 + AM^2 = AP^2.$$

It must be observed that $(\sin A)^2$ is often written for shortness thus, $\sin^2 A$; similarly $(\sin A)^3$ is written thus, $\sin^3 A$. The same mode of abbreviation is used for the powers of the other Trigonometrical functions, and so the result obtained in Art. 32 is usually written thus,

$$\sin^2 A + \cos^2 A = 1.$$

34. *To prove that*

$(\sec A)^2 = 1 + (\tan A)^2$, *and* $(\operatorname{cosec} A)^2 = 1 + (\cot A)^2$.

In the right-angled triangle APM we have

$$AP^2 = PM^2 + AM^2;$$

therefore $$\frac{AP^2}{AM^2} = \frac{PM^2}{AM^2} + 1,$$

therefore $$\left(\frac{AP}{AM}\right)^2 = \left(\frac{PM}{AM}\right)^2 + 1,$$

that is $$(\sec A)^2 = 1 + (\tan A)^2.$$

Again, since $AP^2 = PM^2 + AM^2$,

$$\left(\frac{AP}{PM}\right)^2 = 1 + \left(\frac{AM}{PM}\right)^2,$$

that is $(\text{cosec}\, A)^2 = 1 + (\cot A)^2.$

The results here obtained are usually written thus,

$$\sec^2 A = 1 + \tan^2 A, \quad \text{cosec}^2 A = 1 + \cot^2 A.$$

35. By means of the relations established in Arts. 31...34 we are able to express all the other Trigonometrical Ratios in terms of any one of them; thus, for example, we will express all the rest in terms of the *sine;*

$$\cos A = \sqrt{(1 - \sin^2 A)}; \text{ (Art. 32)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{(1 - \sin^2 A)}}; \text{ (Arts. 31, 32)},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{(1 - \sin^2 A)}}{\sin A}; \text{ (Arts. 31, 32)},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{(1 - \sin^2 A)}}; \text{ (Arts. 31, 32)},$$

$$\text{cosec}\, A = \frac{1}{\sin A}; \text{ (Art. 31)},$$

$$\text{vers}\, A = 1 - \cos A = 1 - \sqrt{(1 - \sin^2 A)}. \text{ (Art. 32)}.$$

Again, we will express all the rest in terms of the tangent;

$$\sin A = \frac{1}{\text{cosec}\, A} = \frac{1}{\sqrt{(1 + \cot^2 A)}} = \frac{1}{\sqrt{\left(1 + \frac{1}{\tan^2 A}\right)}} = \frac{\tan A}{\sqrt{(1 + \tan^2 A)}};$$

(Arts. 31, 34),

$$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{(1 + \tan^2 A)}}; \text{ (Arts. 31, 34)},$$

$$\cot A = \frac{1}{\tan A}; \text{ (Art. 31)},$$

$$\sec A = \sqrt{(1 + \tan^2 A)};$$

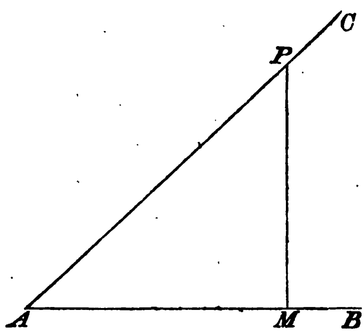
$$\operatorname{cosec} A = \frac{\sqrt{(1+\tan^2 A)}}{\tan A};$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{1}{\sqrt{(1+\tan^2 A)}}.$$

We shall now proceed to determine the values of the Trigonometrical Ratios for some specific angles.

36. *To determine the values of the Trigonometrical Ratios for an angle of* 45°.

Let BAC be an angle of 45°; take any point P in AC and



draw PM perpendicular to AB. Since PAM is half a right angle APM is also half a right angle; therefore $PM = AM$.

Now $$PM^2 + AM^2 = AP^2;$$

thus $$2PM^2 = AP^2;$$

therefore $$\left(\frac{PM}{AP}\right)^2 = \frac{1}{2},$$

therefore $$\frac{PM}{AP} = \frac{1}{\sqrt{2}}.$$

Thus $$\sin 45° = \frac{PM}{AP} = \frac{1}{\sqrt{2}}; \ \cos 45° = \frac{AM}{AP} = \frac{1}{\sqrt{2}};$$

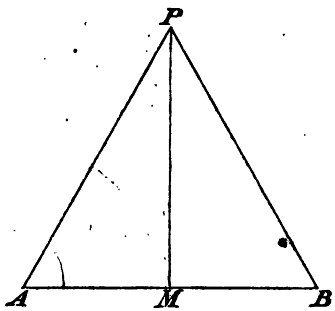
$$\tan 45° = \frac{PM}{AM} = 1; \ \cot 45° = \frac{AM}{PM} = 1;$$

$$\sec 45^\circ = \frac{AP}{AM} = \sqrt{2}; \; \text{cosec}\, 45^\circ = \frac{AP}{PM} = \sqrt{2};$$

$$\text{vers}\, 45^\circ = 1 - \cos 45^\circ = 1 - \frac{1}{\sqrt{2}}.$$

37. *To determine the values of the Trigonometrical Ratios for an angle of* 60° *and for an angle of* 30°.

Let APB be an equilateral triangle, so that the angle PAB



contains 60 degrees; draw PM perpendicular to AB, then $AM = MB$; therefore $AM = \frac{1}{2} AB = \frac{1}{2} AP$.

Thus $\cos 60^\circ = \dfrac{AM}{AP} = \dfrac{1}{2};$

$$\sin 60^\circ = \sqrt{(1 - \cos^2 60^\circ)} = \sqrt{\left(1 - \frac{1}{4}\right)} = \sqrt{\left(\frac{3}{4}\right)} = \frac{\sqrt{3}}{2};$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3};$$

$$\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}};$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2;$$

$$\text{cosec}\, 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}};$$

$$\text{vers } 60^\circ = 1 - \cos 60^\circ = \frac{1}{2}.$$

And $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$; $\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$;

$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}};\ \cot 30^\circ = \tan 60^\circ = \sqrt{3};$$

$$\sec 30^\circ = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}};\ \operatorname{cosec} 30^\circ = \sec 60^\circ = 2;$$

$$\text{vers } 30^\circ = 1 - \cos 30^\circ = 1 - \frac{\sqrt{3}}{2}.$$

38. It may be observed that if an angle be less than 45° the cosine of the angle is *greater* than the sine, and if the angle be greater than 45° and less than 90° the cosine is less than the sine; these results follow immediately from the triangle PAM (see figure in Art. 26) since the greater side in a triangle is opposite to the greater angle.

EXAMPLES.

1. The sine of a certain angle is $\frac{3}{5}$; find the other trigonometrical functions of the angle.

2. The tangent of a certain angle is $\frac{4}{3}$; find the other trigonometrical functions of the angle.

3. The cosine of a certain angle is $\sqrt{\frac{2}{3}}$; find the other trigonometrical functions of the angle.

4. Shew that $\sin^2\theta \tan\theta + \cos^2\theta \cot\theta + 2\sin\theta\cos\theta$
$$= \tan\theta + \cot\theta.$$

5. Shew that $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$

Obtain solutions of the following equations:

6. $\sin^2\theta = \frac{3}{2}\cos\theta$. 7. $\sin\theta + \cos\theta = 1$.

8. $\cot\theta = 2\cos\theta$. 9. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$.

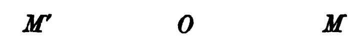
10. $3\sec^4\theta + 8 = 10\sec^2\theta$.

11. Given $\sin(A - B) = \frac{1}{2}$, and $\cos(A + B) = \frac{1}{2}$, find A and B.

IV. APPLICATION OF ALGEBRAICAL SIGNS.

39. In the preceding chapter we defined the Trigonometrical Ratios, and established certain relations between them; we confined ourselves to angles not exceeding a right angle. We shall now extend the definitions so as to render them applicable to angles of any magnitude; the relations which were established will then also be found to be true for angles of any magnitude.

40. Let O be a fixed point in a fixed line, and suppose we have to determine the positions of other points in this line with

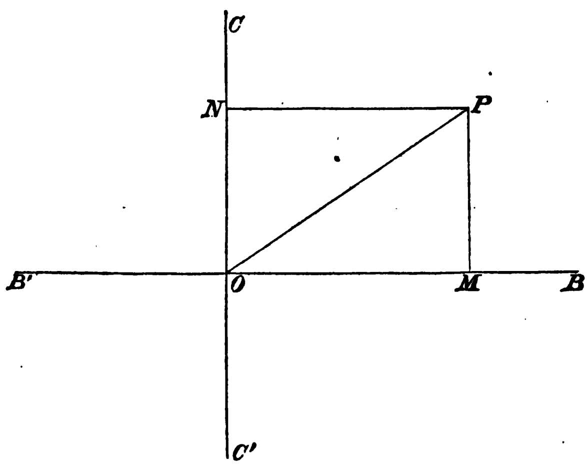


respect to O. The position of any point in the line will be known if we know the distance of the point from O, and also know *on which side of* O *the point lies.* Now it is found convenient to adopt the following *convention;* distances measured in one direction from O along the fixed line will be denoted by *positive* numbers, and distances measured in the opposite direction from O will be denoted by *negative* numbers. Thus, for example, suppose that distances measured from O towards the *right hand* are denoted by *positive* numbers, and let M be a point the distance of which from O is denoted by 2 or $+2$; then if M' be as far from O as M is and on the other side of O, the distance of M' from O will be denoted by -2.

41. We have called this method of determining position by means of numbers affected with algebraical signs a *convention;* we mean by this word to indicate that it is not absolutely *necessary* to adopt this method, but merely *convenient.* The symbols + and − are defined in the beginning of elementary works on Algebra as indicative of the *operations* of addition and subtraction respectively. As the student advances in Algebra he finds that the symbols + and − are also used as indicative of the *qualities* of quantities; and that no contradiction or confusion ultimately arises from this double mode of considering the symbols, but that Algebra gains thereby considerably in power. (See *Algebra,* Chaps. V. and XIV.)

It may be remarked, that we are at liberty to take *either* of the two directions from O as that which will be indicated by *positive* numbers; but when the selection has been made, we must adhere to it throughout the investigations on which we may be engaged.

42. Let OB, OC be two lines which meet at right angles; pro-

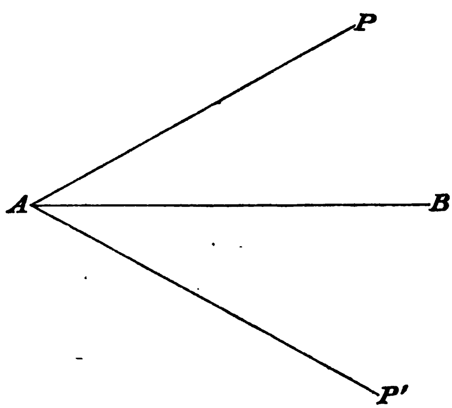


duce BO to any point B' and CO to any point C'. Let P be any point in the plane containing the two lines. The position of P will be known if we know the distance of P from each of the lines

BB' and CC', and also know *on which side* of each of these lines it is situated. Draw PM and PN perpendicular to the lines BB' and CC' respectively. We shall adopt the following conventions; the distance ON or PM will be expressed by a *positive* number when P is *above* the line BB', and by a *negative* number when P is *below* the line BB'; the distance OM or PN will be expressed by a *positive* number when P is to the *right* of CC', and by a *negative* number when P is to the *left* of CC'.

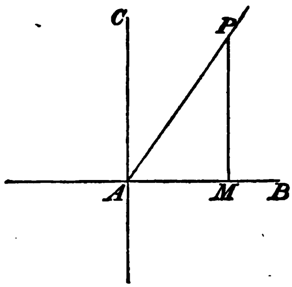
43. A similar convention may conveniently be adopted with respect to *angular* magnitude.

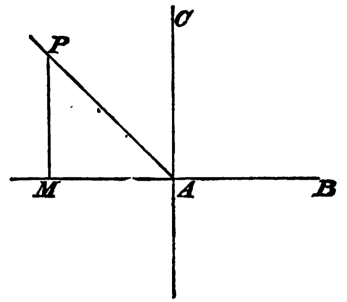
Let a line AP start from the position AB, and by revolving in one direction round A trace out the angle PAB, and let this angle be denoted by a *positive* number; then if the line AP start from the position AB and by revolving round A in the *opposite* direction trace out the angle $P'AB$, this angle may be denoted by a *negative* number. If, for example, each of the angles BAP and BAP' is one-third of a right angle, and we denote the former by the

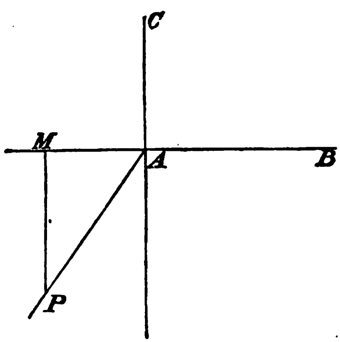


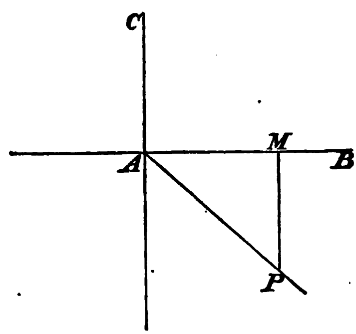
positive fraction $\frac{\pi}{6}$, the latter may be denoted by the negative fraction $-\frac{\pi}{6}$.

44. We shall now give our extended definitions of the Trigonometrical Ratios.









Let AB, AC be two lines at right angles; let a line revolve round the point A from AB towards AC and come into any position AP; draw PM perpendicular to AB or AB produced. Then consider AP always as positive; consider AM as positive or negative according as M is on the same side of AC as B is, or on the opposite side; and consider PM as positive or negative according as P is on the same side of AB as C is, or on the opposite side. Let the angle PAB be denoted by A, then

$$\sin A = \frac{PM}{AP}, \quad \tan A = \frac{PM}{AM}, \quad \sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP}, \quad \cot A = \frac{AM}{PM}, \quad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \quad \operatorname{covers} A = 1 - \sin A.$$

Thus the Trigonometrical Ratios are always whole numbers or fractions positive or negative.

We have therefore Trigonometrical Ratios for any *positive* angle whatever may be its magnitude; and we have also Trigonometrical Ratios for any *negative* angle by adopting the convention that the Trigonometrical Ratios for any negative angle shall be the same as they would be for what we may call the *corresponding* positive angle. Thus, for example, in the last figure we may consider BAP as a negative angle, the magnitude of which is $-\frac{\pi}{3}$; then the Trigonometrical Ratios will be the same as for the angle formed by revolving the moveable line AP in the positive direction until it reaches the position which it has in the figure; so that the Trigonometrical Ratios for the angle $-\frac{\pi}{3}$ will be the same as for the angle $2\pi - \frac{\pi}{3}$.

45. It follows immediately from the definitions, that if two angles differ by 4 right angles or by any multiple of 4 right angles the Trigonometrical Ratios of the two angles are the same.

46. The following relations which have been already established for angles not exceeding a right angle, will now be seen in like manner to hold universally whatever be the magnitude of an angle positive or negative.

$$\tan A \times \cot A = 1, \quad \sec A \times \cos A = 1, \quad \operatorname{cosec} A \times \sin A = 1,$$

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1, \quad \sec^2 A = 1 + \tan^2 A, \quad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

It must be observed that from such an equation as

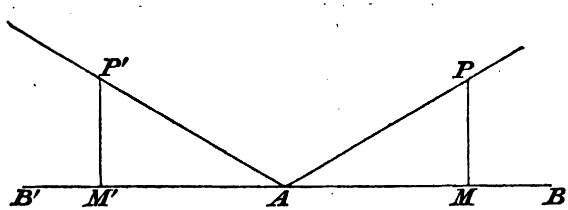
$$\sin^2 A + \cos^2 A = 1,$$

we can infer only that $\sin A = \pm \sqrt{(1 - \cos^2 A)}$, or that $\cos A = \pm \sqrt{(1 - \sin^2 A)}$; we shall have to determine in any particular case *which sign must be ascribed to the radical.*

47. The supplement of an angle is its defect from two right angles. Thus if A denote the *number of degrees* in any angle, $180 - A$ is the *number of degrees* in its supplement; if θ be the *circular measure* of an angle, $\pi - \theta$ is the *circular measure* of its supplement. The verbal definition of the word supplement might appear to limit the word to the case in which the original angle is a positive angle less than two right angles; but the word is used in a wider sense, so that if A be *any* number positive or negative, the angle denoted in degrees by $180 - A$ is called the supplement of that denoted in degrees by A. Similarly, *whatever* θ may be, the angle whose circular measure is $\pi - \theta$, is called the supplement of that whose circular measure is θ.

48. *To compare the Trigonometrical Ratios of any angle and its supplement.*

Let PAB be any angle, produce BA to B' and make $P'AB' = PAB$;



take $AP' = AP$, and draw PM and $P'M'$ perpendicular to BB'.

The angle $P'AB = 180^\circ - P'AB' = 180^\circ - PAB$; thus $P'AB$ is the supplement of PAB. The triangles PAM and $P'AM'$ are geometrically equal in all respects; now

$$\sin A = \frac{PM}{AP}, \qquad \sin (180^\circ - A) = \frac{P'M'}{AP'};$$

and since PM and $P'M'$ are equal in magnitude and *of the same sign*, we have

$$\sin A = \sin (180^\circ - A).$$

Also $$\cos A = \frac{AM}{AP}, \quad \cos (180^\circ - A) = \frac{AM'}{AP'};$$

now AM and AM' are equal in magnitude, but since they are measured in opposite directions from A, they are of *opposite sign;* thus

$$\cos A = -\cos (180^\circ - A).$$

The other Trigonometrical Ratios of the angle A may be compared with those of the supplement either by direct use of the figure, or by employing the two results already established; thus, adopting the latter method,

$$\tan (180^\circ - A) = \frac{\sin (180^\circ - A)}{\cos (180^\circ - A)} = \frac{\sin A}{-\cos A} = -\tan A,$$

$$\cot (180^\circ - A) = \frac{\cos (180^\circ - A)}{\sin (180^\circ - A)} = \frac{-\cos A}{\sin A} = -\cot A,$$

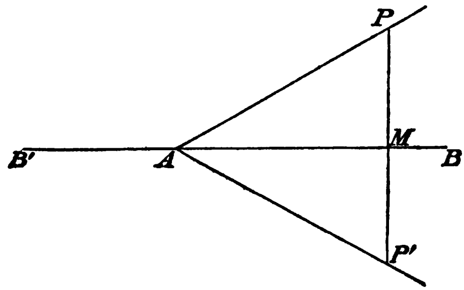
$$\sec (180^\circ - A) = \frac{1}{\cos (180^\circ - A)} = \frac{1}{-\cos A} = -\sec A,$$

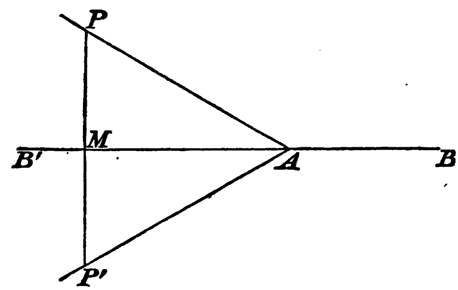
$$\operatorname{cosec} (180^\circ - A) = \frac{1}{\sin (180^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

$$\operatorname{vers} (180^\circ - A) = 1 - \cos (180^\circ - A) = 1 + \cos A.$$

Thus the sine and the cosecant of any angle are respectively the same as the sine and cosecant of the supplement of the angle; all the other Trigonometrical Ratios of any angle, except the versed sine, are *numerically* equal to the corresponding Ratios of the supplement of the angle, but are of *opposite sign.*

49. *To prove that* $\sin(-A) = -\sin A$ *and* $\cos(-A) = \cos A$.





Let PAB be any angle; draw PM perpendicular to BAB', and produce it to P' so that MP' may be equal in length to MP, and join AP'. Then the angles $P'AB$ and PAB which are measured in opposite directions from AB are numerically equal, and if PAB be denoted by A, then $P'AB$ will be denoted by $-A$. And

$$\sin A = \frac{PM}{AP}, \quad \sin(-A) = \frac{P'M}{AP'};$$

and $P'M$ is numerically equal to PM, but of opposite sign; thus

$$\sin(-A) = -\sin A.$$

Also
$$\cos(-A) = \frac{AM}{AP'} = \frac{AM}{AP} = \cos A.$$

Moreover,
$$\tan(-A) = \frac{\sin(-A)}{\cos(-A)} = \frac{-\sin A}{\cos A} = -\tan A;$$

$$\cot(-A) = \frac{\cos(-A)}{\sin(-A)} = \frac{\cos A}{-\sin A} = -\cot A;$$

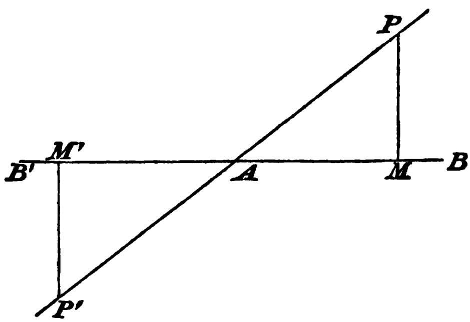
$$\sec(-A) = \frac{1}{\cos(-A)} = \frac{1}{\cos A} = \sec A;$$

$$\operatorname{cosec}(-A) = \frac{1}{\sin(-A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A;$$

$$\operatorname{vers}(-A) = 1 - \cos(-A) = 1 - \cos A = \operatorname{vers} A.$$

50. *To prove that* $\sin(180^\circ + A) = -\sin A$ *and* $\cos(180^\circ + A) = -\cos A$.

Let PAB be any angle, produce PA to P' so that AP' may be equal in length to AP. Draw PM and $P'M'$ perpendicular to



BAB'. Then if PAB be denoted by A, the angle $P'AB$ measured in the same direction from AB will be denoted by $180^\circ + A$.

The triangles PAM and $P'AM'$ are geometrically equal in all respects;

and
$$\sin A = \frac{PM}{AP}, \quad \sin(180^\circ + A) = \frac{P'M'}{AP'};$$

$$\cos A = \frac{AM}{AP}, \quad \cos(180^\circ + A) = \frac{AM'}{AP'}.$$

Now PM and $P'M'$ are numerically equal but of opposite sign; also AM and AM' are numerically equal but of opposite sign; thus

$$\sin(180^\circ + A) = -\sin A, \quad \cos(180^\circ + A) = -\cos A;$$

moreover $\tan(180^\circ+A)=\dfrac{\sin(180^\circ+A)}{\cos(180^\circ+A)}=\dfrac{-\sin A}{-\cos A}=\tan A,$

$$\cot(180^\circ+A)=\frac{\cos(180^\circ+A)}{\sin(180^\circ+A)}=\frac{-\cos A}{-\sin A}=\cot A;$$

similarly $\sec(180^\circ+A)=-\sec A,\quad \operatorname{cosec}(180^\circ+A)=-\operatorname{cosec} A.$

It is obviously only another mode of expressing the two fundamental results if we write

$$\sin A=-\sin(A-180^\circ),\quad \cos A=-\cos(A-180^\circ).$$

51. The results of Arts. 48, 49, and 50, are true whatever be the magnitude of the angle A, and whether A be positive or negative. This the student should carefully notice. First consider Art. 49; *whatever the magnitude of A may be, positive or negative*, we shall always have PMP' forming a straight line, and the points P and P' equally distant from M and on opposite sides of it; and the angles PAB and $P'AB$ will be numerically equal but of opposite sign. Thus we become certain of the universal truth of Art. 49. Next consider Art. 50; the essential points of the demonstration are that M and M' should be equally distant from A and on opposite sides of it, and that P and P' should be equally distant from the line BAB' and on opposite sides of it; and the figure assures us that these essential points are always secured. If PAB be *any* positive angle, then by adding to it an angle of 180° we obtain the angle formed by AB and AP'. If $P'AB$ be *any* negative angle, then by adding to it an angle of 180° we obtain the angle formed by AP and AB. Thus we become certain of the universal truth of Art. 50. The universal truth of Art. 48 may be made to depend on that of Art. 49 and that of Art. 50. For we have

$\sin A=-\sin(A-180^\circ)$, universally, by Art. 50,

$\sin(A-180^\circ)=-\sin(180^\circ-A)$, universally, by Art. 49,

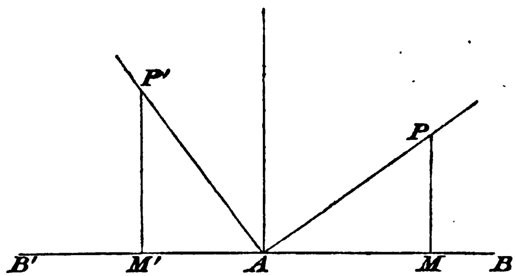
therefore $\sin A=\sin(180^\circ-A)$ universally.

Again $\cos A=-\cos(A-180^\circ)$, universally, by Art. 50,

$\cos(A - 180^\circ) = \cos(180^\circ - A)$, universally, by Art. 49, therefore $\cos A = -\cos(180^\circ - A)$, universally.

52. *To shew that* $\sin(90^\circ + A) = \cos A$,

and $\cos(90^\circ + A) = -\sin A$.



Let PAB be any angle; let AP' be at right angles to AP and so situated that a moveable line can pass from the position AP to the position AP' by revolving round A in the *positive direction* through a right angle. Then if PAB be denoted by A we can denote $P'AB$ by $90^\circ + A$. Take $AP' = AP$ and draw PM and $P'M'$ perpendicular to BAB'. Then the angle PAM is geometrically equal to the angle $AP'M'$, and the triangles PAM and $P'AM'$ are geometrically equal in all respects. And

$$\sin(90^\circ + A) = \frac{P'M'}{AP'}, \quad \cos A = \frac{AM}{AP};$$

now $P'M'$ is numerically equal to AM and both are of the same sign (Art. 42); thus

$$\sin(90^\circ + A) = \cos A.$$

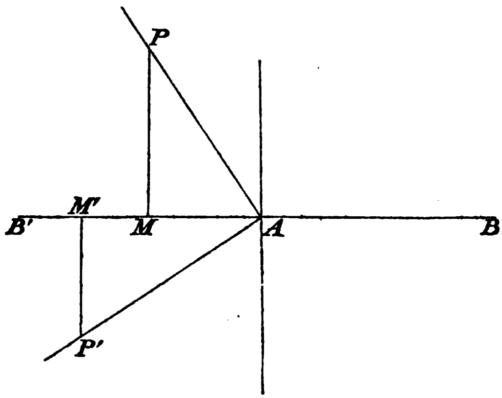
Again $$\cos(90^\circ + A) = \frac{AM'}{AP'}, \quad \sin A = \frac{PM}{AP};$$

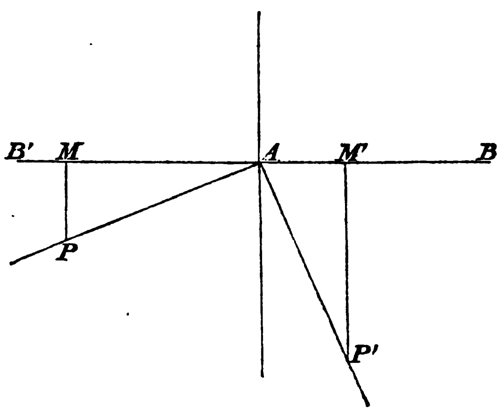
now AM' and PM are numerically equal but of opposite sign (Art. 42); thus $\cos(90^\circ + A) = -\sin A$.

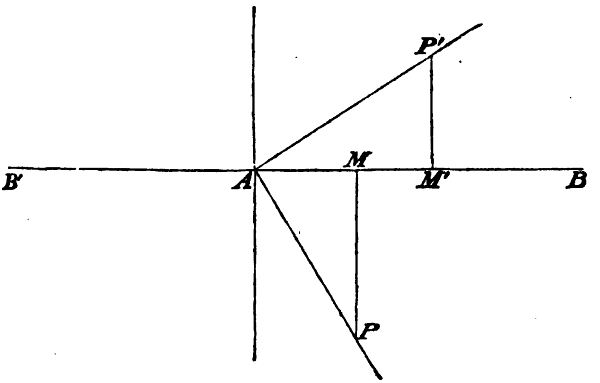
53. In order to prove that the proposition in the preceding article is universally true, we must examine the different cases

that can occur; the figure in the preceding article supposes that A is a positive angle terminated in the first quadrant. The annexed three figures shew AP in the second, third, and fourth quadrants respectively.

In every case it will be seen that the triangles PAM and $P'AM'$ are geometrically equal in all respects; also $P'M'$ and AM are of the *same* sign, and AM' and PM are of *opposite* sign. Thus the proposition may be seen to be true if A be any *positive* angle.







The four figures of this and the preceding article will also shew the truth of the proposition for any *negative* angle; the last figure for example applies when A is between 0 and -90°, the third figure when A is between -90° and -180°, the second figure when A is between -180° and -270°, and the first figure when A is between -270° and -360°.

54. If A be the number of degrees in *any* angle, then the angle which is expressed in degrees by $90 - A$ is called the *complement* of the angle A; so $\frac{\pi}{2} - \theta$ is the circular measure of the *complement* of the angle whose circular measure is θ. The term *complement* of an angle has already been introduced (Art. 29), but the angle contemplated then was a positive angle less than a right angle. This restriction however will be no longer retained. We may now shew universally that the *sine of an angle is equal to the cosine of its complement, and the cosine of an angle equal to the sine of its complement.* These propositions may be proved by examining different cases as in Arts. 52 and 53; or they may be deduced from results already established. Thus, for example, we have proved that

$\sin(90^\circ + A) = \cos A$, universally (Arts. 52, 53),

also $\sin(90^\circ + A) = \sin(180^\circ - 90^\circ - A)$, universally (Art. 51),

therefore $\sin(90^\circ - A) = \cos A$, universally.

Then if we suppose $90^\circ - A = A'$ we have $A = 90^\circ - A'$; thus

$$\sin A' = \cos(90^\circ - A'), \text{ universally.}$$

55. It will now be found that we are able to express the Trigonometrical Ratios of any angle whatever in terms of the Trigonometrical Ratios of some positive angle not exceeding a right angle. For in the first place by the formulæ $\sin(-A) = -\sin A$ and $\cos(-A) = \cos A$, and those which follow from these (see Art. 49), we can make the Trigonometrical Ratios of any *negative* angle depend upon those of the corresponding *positive* angle; and so we need only consider positive angles if we please. By Art. 45 any multiple of four right angles may be rejected; thus, so far as its Trigonometrical Ratios are concerned, we may replace any angle whatever by an angle less than four right angles. Then by the formulæ $\sin(180^\circ + A) = -\sin A$, and $\cos(180^\circ + A) = -\cos A$, and those which follow from these (see Art. 50), we may make the Trigonometrical Ratios of any angle depend upon those of an angle not exceeding two right angles. Lastly, by the formulæ $\sin(180^\circ - A) = \sin A$ and $\cos(180^\circ - A) = -\cos A$, and those which follow from these (see Art. 48), we may make the Trigonometrical Ratios of any angle depend upon those of an angle not exceeding a right angle.

For example,

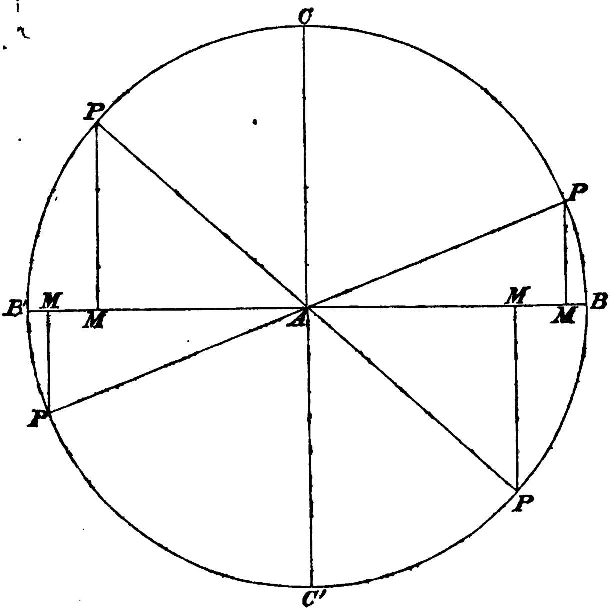
$$\sin 600^\circ = \sin(360^\circ + 240^\circ) = \sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ.$$

$$\text{Tan}(-1000^\circ) = -\tan 1000^\circ = -\tan(720^\circ + 280^\circ) = -\tan 280^\circ$$
$$= -\tan(180^\circ + 100^\circ) = -\tan 100^\circ = -\tan(180^\circ - 80^\circ) = \tan 80^\circ.$$

56. *To trace the changes in the sine of an angle as the angle varies.*

Let BAB' and CAC' be two lines at right angles, and suppose a line AP of constant length to revolve round one end A from the fixed position AB so that P traces out the circle $BCB'C'$. From any position of P draw PM perpendicular to BAB'; then

$$\sin PAB = \frac{PM}{AP}.$$



When AP coincides with AB the perpendicular PM vanishes; thus when the angle is zero so also is its sine. While AP moves through the first quadrant PM is positive, and continually increases until AP coincides with AC, and then PM is equal to AP; thus as the angle increases from 0 to 90° the sine increases from 0 to 1. While AP moves through the second quadrant PM is positive, and continually decreases until AP coincides with AB' and then PM vanishes; thus as the angle increases from 90° to 180° the sine diminishes from 1 to 0. While AP moves through the third quadrant PM is negative, and increases *numerically* until AP coincides with AC'; thus as the angle increases from 180° to 270° the sine is *negative* and increases numerically from 0 to -1. While AP moves through the fourth quadrant PM is negative, and decreases *numerically* until AP coincides with AB; thus as the angle increases from 270° to 360° the sine is *negative* and decreases numerically from -1 to 0.

57. *To trace the changes in the cosine of an angle as the angle varies.*

With the figure of the preceding article we have

$$\cos PAB = \frac{AM}{AP}.$$

At first AP coincides with AB and then $AM = AP$; thus when the angle is zero the cosine is 1. While AP moves through the first quadrant AM is positive and continually decreases until AP coincides with AC and then AM vanishes; thus as the angle increases from 0 to 90° the cosine diminishes from 1 to 0. While AP moves through the second quadrant AM is negative and increases *numerically* until AP coincides with AB'; thus as the angle increases from 90° to 180° the cosine is *negative* and increases numerically from 0 to -1. While AP moves through the third quadrant AM is negative and decreases *numerically* until AP coincides with AC'; thus as the angle increases from 180° to 270° the cosine is *negative* and decreases numerically from -1 to 0. While AP moves through the fourth quadrant AM is positive and continually increases until AP coincides with AB; thus as the angle increases from 270° to 360° the cosine is positive and increases from 0 to 1.

58. *To trace the changes in the tangent of an angle as the angle varies.*

With the figure of Art. 56 we have

$$\tan PAB = \frac{PM}{AM}.$$

At first AP coincides with AB and then PM vanishes and $AM = AB$; thus when the angle is zero so also is its tangent. While AP moves through the first quadrant PM and AM are positive; PM continually increases and AM continually decreases until AP coincides with AC; thus as the angle increases from 0 to 90° the tangent increases from 0 without limit, so that by taking an angle sufficiently near to 90° we can make the tangent as great as we please; this is usually expressed for the sake of abbreviation thus, *the tangent of* 90° *is infinite.* While AP moves through the second quadrant PM is positive and AM is negative; PM continually decreases and AM increases *numerically* until AP coincides with AB'; thus as the angle increases from 90° to 180° the tangent is *negative* and decreases numerically from an indefinitely large value to zero. While AP moves through the third quadrant PM and AM are negative; PM increases *numerically* and AM decreases *numerically* until AP coincides with AC'; thus as the angle increases from 180° to 270° the tangent is positive and increases from 0 without limit, so that by taking an angle sufficiently near to 270° we can make the tangent as great as we please; this as before is abbreviated into *the tangent of* 270° *is infinite.* While AP moves through the fourth quadrant PM is negative and AM is positive; PM continually decreases numerically and AM increases until AP coincides with AB; thus as the angle increases from 270° to 360° the tangent is *negative* and decreases numerically from an indefinitely large value to zero.

Similarly the changes in the cotangent of an angle may be traced.

59. *To trace the changes in the secant of an angle as the angle varies.*

The changes in the secant of an angle may be traced by means of the figure in the same way as those of the sine, cosine, and tangent; or we may use the formula $\sec PAB = \frac{1}{\cos PAB}$, and infer the changes in the secant from the known changes in the cosine; we will adopt the latter method. As the angle increases from 0 to 90° the cosine diminishes from 1 to 0; thus the secant increases from 1 without limit, so we may say *the secant of* 90° *is infinite.* As the angle increases from 90° to 180° the cosine is *negative* and increases *numerically* from 0 to -1; thus the secant is *negative* and decreases *numerically* from an indefinitely large value to -1. As the angle increases from 180° to 270° the cosine is *negative* and decreases *numerically* from -1 to 0; thus the secant is *negative* and increases numerically from -1 to infinity. As the angle increases from 270° to 360° the cosine is positive and continually increases from 0 to 1; thus the secant is positive and continually diminishes from infinity to 1.

Similarly the changes in the cosecant of an angle may be traced.

60. Since vers $A = 1 - \cos A$, as the angle increases from 0 to 180° the versed sine increases from 0 to 2, and as the angle increases from 180° to 360° the versed sine diminishes from 2 to 0.

61. Thus we see that the sine and cosine may have any value between -1 and $+1$; the tangent and cotangent may have any value between $-\infty$ and $+\infty$; the secant and cosecant may have any value between $-\infty$ and -1 and between $+1$ and $+\infty$. And it will be found on examination that no Trigonometrical Ratio changes its sign except when it passes through the value zero or the value infinity. The versed sine is always positive and may have any value between 0 and 2.

62. The following table of the values of the Trigonometrical Ratios of certain angles is formed from the results of the preceding chapter and the present chapter.

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

EXAMPLES.

1. Determine the values of the Trigonometrical ratios for an angle of 585°.

2. Also for an angle of 690°.

3. Also for an angle of 930°.

4. Also for an angle of 6420°.

5. Find all the angles between 0 and 900° which satisfy the relation $\tan\theta = 1$.

6. Find all the angles between 0 and 900° which satisfy the relation $\cos^2\theta = \frac{1}{2}$.

7. Find all the values of versin $\frac{n\pi}{4}$ where n is any integer.

8. Find all the values of $\sin\left\{\frac{n\pi}{2} + (-1)^n\frac{\pi}{6}\right\}$ where n is any integer.

9. Solve $\sin^2\theta + \cos^2\theta = 0$.

10. Solve $2\sin^2\theta - 5\cos\theta - 4 = 0$.

11. Trace the changes in the sign and value of $\cos\theta - \sin\theta$ as θ changes from 0 to 2π.

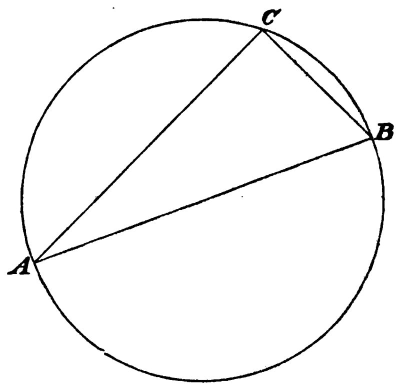
12. Also of $\cos^2\theta - \sin^2\theta$.

13. Also of $\tan\theta + \cot\theta$.

14. Is $\sec^2\theta = \frac{4ab}{(a+b)^2}$ a possible equation?

V. ANGLES WITH GIVEN TRIGONOMETRICAL RATIOS.

63. *To construct an angle with a given sine or cosine.*



Required an angle the *sine* of which is a given quantity a. Describe a circle with unity for its diameter, and take any diameter AB of this circle; with centre B and radius a describe a circle; let C be one of the points where this circle meets the former circle; join AC and BC. Then ACB is a right angle, and the sine of BAC is $\frac{BC}{AB}$, that is a; therefore BAC is such an angle as is required.

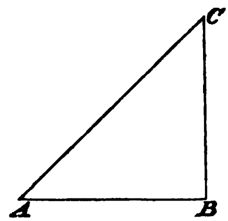
If the *cosine* of the required angle is to be a, then the same construction may be made, and ABC will be such an angle as is required.

64. *To construct an angle with a given tangent or cotangent.*

Required an angle the *tangent* of which is a given quantity a.

Take a line AB the length of which is unity; draw BC at right angles to AB and equal in length to a, and join CA. Then the tangent of BAC is $\frac{BC}{BA}$, that is a; therefore BAC is such an angle as is required.

If the *cotangent* of the required angle is to be a then the same construction may be made, and ACB will be such an angle as is required.

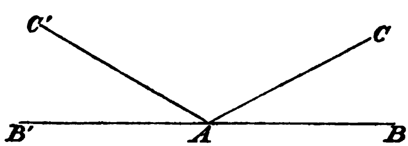


65. If an angle is required to have a given cosecant, then since the cosecant is the reciprocal of the sine, the angle must have a known *sine;* therefore the angle may be found by Art. 63. Similarly if an angle is required to have a given secant, or a given versed sine, then the cosine of the angle is known and the angle may be found by Art. 63.

We shall now proceed to find expressions which include all the angles which have a given Trigonometrical Ratio. In the remainder of this chapter we shall express all the angles that occur in *circular measure.*

66. *To find an expression for all the angles which have a given sine.*

Let BAC be the least positive angle which has the given sine;



denote this angle by α. Produce BA to any point B' and make the angle $B'AC' = BAC$; then $BAC' = \pi - \alpha$.

Now it is obvious from the figure that the only *positive* angles which have the same sine as α are $\pi - \alpha$, and the angles formed by adding any multiple of four right angles to α or to $\pi - \alpha$; that is, angles included in the formulæ $2n\pi + \alpha$ and $2n\pi + \pi - \alpha$, where n is zero or any positive integer. Also the only *negative* angles which have the same sine as a are $-(\pi + a)$, and $-(2\pi - a)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is angles included in the formulæ $2n\pi - (\pi + a)$, and $2n\pi - (2\pi - a)$ where n is zero or any negative integer. All the angles which have been indicated will be found on trial to be included in the formula

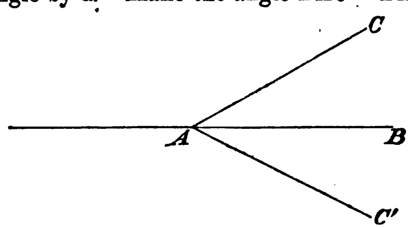
$$n\pi + (-1)^n a,$$

where n is zero, or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated. Thus the formula $n\pi + (-1)^n a$ includes all the angles which have the same sine as a, and all the angles which it includes have the same sine as a.

This formula also determines all the angles which have the same cosecant as a.

67. *To find an expression for all the angles which have a given cosine.*

Let BAC be the least positive angle which has the given cosine; denote this angle by a. Make the angle $BAC' = BAC$. Now it is



obvious from the figure, that the only *positive* angles which have the same cosine as a are $2\pi - a$, and the angles formed by adding any multiple of four right angles to a or to $2\pi - a$; that is, angles included in the formulæ $2n\pi + a$ and $2n\pi + 2\pi - a$, where n is zero or any positive integer. Also the only negative angles which have the same cosine as a are $-a$, and $-(2\pi - a)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the formulæ $2n\pi - a$ and $2n\pi - (2\pi - a)$ where n is zero or any negative integer. All the angles which have been indicated will be found on trial to be included in the formula

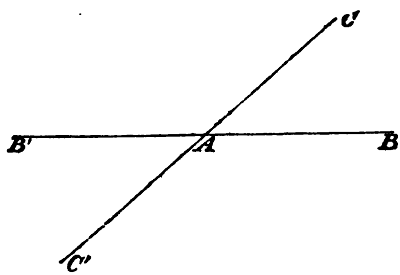
$$2n\pi \pm a,$$

where n is zero or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated. Thus the formula $2n\pi \pm a$ includes all the angles which have the same cosine as a, and all the angles which it includes have the same cosine as a.

This formula also determines all the angles which have the same secant or the same versed sine as a.

68. *To find an expression for all the angles which have a given tangent.*

Let BAC be the least positive angle which has the given tangent; denote this angle by a. Produce BA to any point B' and CA to any point C'.



Now it is obvious from the figure that the only *positive* angles which have the same tangent as a are $\pi + a$, and the angles formed by adding any multiple of four right angles to a or to $\pi + a$; that is, angles included in the formulæ $2n\pi + a$ and $2n\pi + \pi + a$, where n is zero or any positive integer. Also the only *negative* angles which have the same tangent as a are $-(\pi - a)$, and $-(2\pi - a)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the formulæ $2n\pi - (\pi - a)$ and $2n\pi - (2\pi - a)$ where n is zero or any negative integer. All the angles which have been indicated will be found on trial to be included in the formula

$$n\pi + a,$$

where n is zero, or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated. Thus the formula $n\pi + \alpha$ includes all the angles which have the same tangent as α, and all the angles which it includes have the same tangent as α.

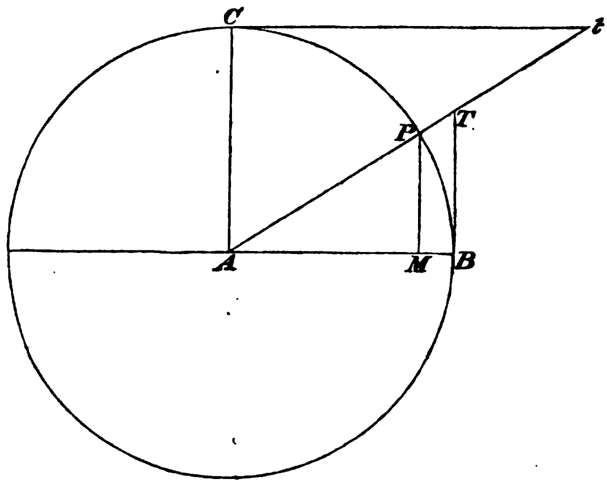
This formula also determines all the angles which have the same *cotangent* as α.

69. In Art. 66 we shewed that if α be the least positive angle which has a given sine, the formula $n\pi + (-1)^n\alpha$ includes without excess or defect *all* the angles which have the same sine as α; it was convenient for distinctness in the demonstration to suppose α the *least positive angle* which has the given sine. But this restriction can be removed, for we can shew that if β be *any* angle, the formula $n\pi + (-1)^n\beta$ will include without excess or defect all the angles which have the same sine as β. For suppose α to be the least positive angle which has its sine equal to $\sin\beta$; then, from what has been proved, we know that β must be *one* of the angles included in the formula $m\pi + (-1)^m\alpha$ where m is zero, or any integer positive or negative. Suppose then $\beta = r\pi + (-1)^r\alpha$; therefore $n\pi + (-1)^n\beta = n\pi + (-1)^n r\pi + (-1)^{n+r}\alpha$; and all we have to prove is, that this formula includes without excess or defect all the angles included in the formula $m\pi + (-1)^m\alpha$. If n be even the formulæ correspond by taking $m = n + r$; if n be odd, the formulæ correspond by taking $m = n - r$. The formula $n\pi + (-1)^n\beta$ will of course also include without excess or defect all the angles which have the same cosecant as β.

70. Similarly we may shew that if β be *any* angle, the angles which have the same cosine or secant or versed sine as β will be included without excess or defect in the formula $2n\pi \pm \beta$; and that the angles which have the same tangent or cotangent as β will be included without excess or defect in the formula $n\pi + \beta$.

71. Before leaving this part of the subject we will recur to the definitions of the Trigonometrical Functions; we considered them

as *ratios* formed by comparing the sides of a right-angled triangle, but formerly they were differently defined, and it is advisable to notice the old definitions in order that the student may understand allusions to them which will occur in his reading.



Let A be the centre of any circle, AB a radius, BP any arc; draw the radius AC at right angles to AB, and draw tangents to the circle at the points B and C; produce AP to meet the first tangent in T and the second tangent in t; draw PM perpendicular to AB. Then the old definitions are as follows, in which the *lines* of the figure are considered to be functions of the arc BP. PM is the sine of the arc BP, AM is its cosine, BT is its tangent, Ct is its cotangent, AT is its secant, At is its cosecant, BM is its versed sine; also the line joining B and P is the *chord* of the arc BP. Thus the terms *sine, cosine*, &c., formerly denoted certain *lines* and not certain *ratios*. On the old system the lengths of the *sine, cosine*, &c. depended on the radius of the circle considered, so that it became necessary to state what length was ascribed to this radius in any investigation.

72. It is easy to connect the values of the old and new Trigonometrical Functions; for

$$\text{sine of the } \textit{angle } PAB = \frac{PM}{AP},$$

$$\text{sine of the } \textit{arc } PB = PM;$$

thus sine of the *arc* = radius of circle × sine of the *angle*,

and $$\text{sine of the } \textit{angle} = \frac{\text{sine of the } \textit{arc}}{\text{radius of circle}}.$$

Similar results hold for all the other Trigonometrical Functions. Thus from any formula in the modern system which involves Functions of *Angles*, we can deduce the corresponding formula in the ancient system which will involve Functions of *arcs*, and *vice versa*.

For example, if A denote any angle, we have (Art. 32)

$$\sin^2 A + \cos^2 A = 1.$$

Now let a denote the arc corresponding to A in a circle of radius r; then, using the old definitions

$$\frac{\sin^2 a}{r^2} + \frac{\cos^2 a}{r^2} = 1,$$

so that $$\sin^2 a + \cos^2 a = r^2.$$

We may notice that the sine of half the angle PAB

$$= \frac{\frac{1}{2}PB}{AB} = \frac{1}{2}\frac{PB}{AB};$$

and therefore the chord of an arc = radius of circle × twice the sine of half the angle.

73. Since the sine of an *arc* is equal to the radius of the circle multiplied by the sine of the angle, it follows that if the *radius of the circle be unity* the numerical value of the sine is the same in *both* systems; and a similar result holds for the other Trigonometrical Functions. Thus any formula expressed in the ancient system may be immediately converted into a formula expressed in the modern system by supposing the *radius of the circle to be equal to unity*.

74. The old definitions give some indications of the origin of the terms *sine*, *cosine*, &c. The word *sine* seems derived from the Latin word *sinus* a bosom, the *arc* is supposed to represent a bow, and thus gets its name, and the string, half of which represents the sine of half the arc, would come against the breast of the archer. The words *tangent* and *secant* are naturally derived from the old definitions. (See *Penny Cyclopædia;* article *Trigonometry.*)

75. The modern method has now completely superseded the ancient method in English works; it was introduced by Dr Peacock. (See Peacock's *Algebra*, Vol. II. p. 157). It may however be observed, that it is stated by Professor De Morgan (*Trigonometry* and *Double Algebra*, p. 18), that "Rheticus, who gave the first complete trigonometrical table, and invented the secant and cosecant to complete it, used the method of ratios."

EXAMPLES.

1. Write down the general value of θ when $\tan\theta = 1$.

2. Write down the general value of θ when $\sin\theta = 1$.

3. Write down the general value of θ when $\cos\theta = 1$.

4. Write down the general value of θ when $\cos\theta = -\frac{1}{2}$.

5. Find all the values of θ which satisfy $\sin^2\theta = \sin^2\alpha$.

6. Write down the general value of θ when $\operatorname{cosec}^2\theta = \frac{4}{3}$.

7. Find all the values of θ which satisfy $\cos^2\theta = \cos^2\alpha$.

8. Write down the general value of θ when $\sec^2\theta = 2$.

9. Find all the values of θ which satisfy $\tan^2\theta = \tan^2\alpha$.

10. Write down the general value of θ when $\tan^2\theta = \frac{1}{3}$.

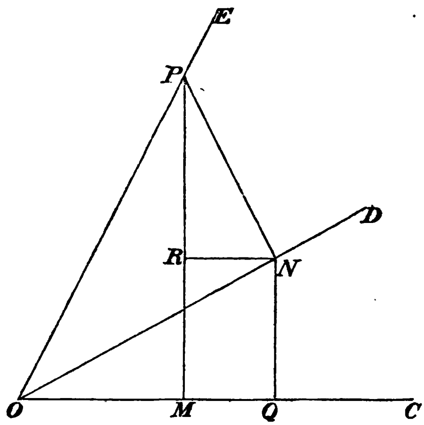
11. Shew that all the angles which have both the same sine and the same cosine as α, are included in the formula $2n\pi + \alpha$.

12. Write down the general value of θ which satisfies *both*

$$\sin\theta = -\frac{1}{2} \text{ and } \cos\theta = -\frac{\sqrt{3}}{2}.$$

VI. TRIGONOMETRICAL RATIOS OF TWO ANGLES.

76. *To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A+B$. In OE take any point P, draw PM perpendicular to OC, and PN perpendicular to OD; draw NR perpendicular to PM and NQ perpendicular to OC. Then the angle PNR is the complement of RNO, that is of NOC; therefore NPR is equal to A.

$$\text{Now } \sin(A+B)=\frac{PM}{OP}=\frac{RM+PR}{OP}=\frac{NQ}{OP}+\frac{PR}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}+\frac{PR}{PN}\cdot\frac{PN}{OP}$$

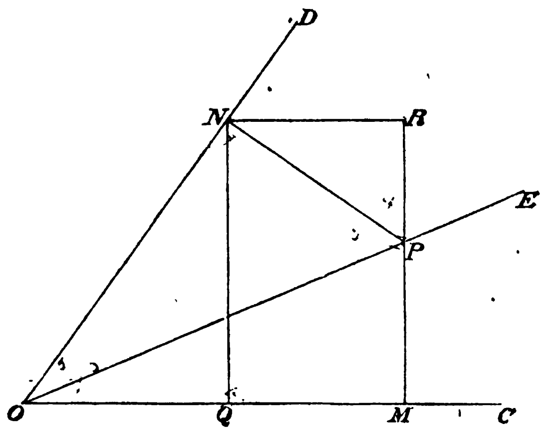
$$=\sin A\cos B+\cos A\sin B.$$

$$\cos(A+B)=\frac{OM}{OP}=\frac{OQ-QM}{OP}=\frac{OQ}{OP}-\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}-\frac{NR}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A\cos B-\sin A\sin B.$$

77. *To express the sine and cosine of the difference of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A-B$. In OE take any point P, draw PM perpendicular to OC and PN perpendicular to OD; draw NR perpendicular to MP produced and NQ perpendicular to OC. Then the angle PNR is the complement of PNQ, and is therefore equal to ONQ; therefore NPR is equal to A.

$$\text{Now } \sin(A-B)=\frac{PM}{OP}=\frac{RM-RP}{OP}=\frac{NQ}{OP}-\frac{RP}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}-\frac{RP}{PN}\cdot\frac{PN}{OP}$$

$$=\sin A\cos B-\cos A\sin B.$$

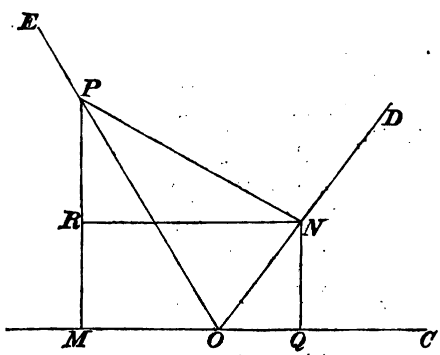
$$\cos(A-B)=\frac{OM}{OP}=\frac{OQ+QM}{OP}=\frac{OQ}{OP}+\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}+\frac{NR}{PN}\cdot\frac{PN}{OP}$$

$$=\cos A\cos B+\sin A\sin B.$$

78. To assist the student in remembering the preceding demonstrations, we may observe that the point P is taken in the line that *bounds the compound angle we are considering;* thus, in proving the formulæ for $\sin(A+B)$ and $\cos(A+B)$ the point P is taken in the line which bounds the angle $A+B$, and in proving the formulæ for $\sin(A-B)$ and $\cos(A-B)$ the point P is taken in the line which bounds the angle $A-B$. After the construction is completed, the principal step consists in shewing that the angle NPR is equal to A; it will be seen from the construction that this is the case, for the lines PN, RP are respectively *perpendicular to the lines which form the angle* A, and thus form an angle equal to A.

79. The formulæ established in Arts. 76 and 77 are true whatever may be the size of the angles A and B; the student may exercise himself by going through the construction and demonstration in different cases; it will be found that the only variety which occurs in the construction consists in the circumstance that the perpendiculars instead of falling upon certain lines may fall upon those lines *produced.* We will, as an example, prove the formulæ in Art. 76, when each of the angles A and B is less than a right angle, and their sum greater than a right angle.



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A+B$. In OE take any point P, draw PM perpendicular to CO produced and PN perpendicular to OD; draw NR perpendicular to PM and NQ perpendicular to OC. Then the angle PNR is the complement of RNO, that is of NOC; therefore NPR is equal to A.

$$\text{Now } \sin(A+B) = \frac{PM}{OP} = \frac{MR+PR}{OP} = \frac{NQ}{OP} + \frac{PR}{OP}$$

$$= \frac{NQ}{ON}\cdot\frac{ON}{OP} + \frac{PR}{PN}\cdot\frac{PN}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

Also $\qquad \cos(A+B) = \dfrac{OM}{OP};$

here we must remember that OM being measured to the *left* of O is a negative quantity, and we may put for it $OQ - QM$, that is $OQ - NR$; thus

$$\cos(A+B) = \frac{OQ-NR}{OP} = \frac{OQ}{OP} - \frac{NR}{OP}$$

$$= \frac{OQ}{ON}\cdot\frac{ON}{OP} - \frac{NR}{PN}\cdot\frac{PN}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

80. The formulæ established in Arts. 76 and 77 may be considered the fundamental formulæ of the subject; it is important therefore that they should be shewn to be universally true. As we have intimated in the preceding article, the student might convince himself of their universal truth by examination of all the cases that can occur; but we may arrive at the required result more decisively by making use of some theorems which have already been completely established.

The formulæ we have to prove are

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ .... (1).

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ .... (2).

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ .... (3).

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ .... (4).

Now in Arts. 76 and 79 we have shewn that (1) and (2) hold for all positive values of A and B, which do not exceed a right angle; and in Art. 77 we have shewn that (3) and (4) hold for all positive values of A and B which do not exceed a right angle, *provided A be greater than B.* We shall first shew that the restriction of A being greater than B may be removed from (3) and (4).

By Art. 49, $\sin(A-B) = -\sin(B-A)$,

and $\cos(A-B) = \cos(B-A)$;

if then we know that

$$\sin(B-A) = \sin B\cos A - \cos B\sin A,$$

and $$\cos(B-A) = \cos B\cos A + \sin B\sin A;$$

we know also that

$$\sin(A-B) = \sin A\cos B - \cos A\sin B,$$

and $$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Therefore if (3) and (4) hold for values of A and B comprised between any limits when A is *greater* than B, they hold for values of A and B comprised between the same limits when A is *less* than B.

Thus we know that the four formulæ are all true for any positive value of each angle between zero and a right angle. We shall next shew that if all the formulæ are true for values of A and B comprised between certain limits, these limits may be increased by a right angle. For by Art. 52,

$$\sin(90^\circ + A + B) = \cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$= \sin(90^\circ + A)\cos B + \cos(90^\circ + A)\sin B;$$

in this way, from the truth of (2) for any limits, we can infer the truth of (1) with an increase of 90° in the limits of either angle. Similar considerations apply to all the other formulæ; and thus the limits become as large as we please.

Lastly, the truth of the formulæ for any negative angles may be established; suppose A and B both negative, let $A = -A'$ and $B = -B'$; thus

$$\sin(A+B) = \sin(-A'-B') = -\sin(A'+B'), \text{ by Art. 49,}$$
$$= -(\sin A' \cos B' + \cos A' \sin B')$$
$$= \sin(-A')\cos(-B') + \cos(-A')\sin(-B')$$
$$= \sin A \cos B + \cos A \sin B.$$

Similarly all the other formulæ may be shewn to be true when both the angles are negative, or when one of the angles is negative.

81. From the four fundamental formulæ a large number of other formulæ may be deduced; we shall give some examples of such deductions.

82. In the expressions for $\sin(A+B)$ and $\cos(A+B)$ put $B=A$; thus

$$\sin 2A = 2 \sin A \cos A;$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$

Thus
$$1 + \cos 2A = 2\cos^2 A,$$
$$1 - \cos 2A = 2\sin^2 A,$$
and
$$\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A.$$

83. From the four fundamental formulæ we have

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B,$$
$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B,$$
$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B,$$
$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B.$$

Let $A+B=C$ and $A-B=D$; therefore

$$A = \tfrac{1}{2}(C+D) \text{ and } B = \tfrac{1}{2}(C-D); \text{ thus}$$

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$
$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$
$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

84. $\text{Sin}(A+B)\sin(A-B)$
$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$
$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$
$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$
$$= \sin^2 A - \sin^2 B.$$

And $\cos(A+B)\cos(A-B)$
$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$$
$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$
$$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

85. $\text{Tan}(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$;

divide both numerator and denominator of the last expression by $\cos A\cos B$; thus we get $\dfrac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$;

therefore $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Suppose $B = A$; thus we obtain
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

$$\text{Tan}(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$$
$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

Suppose for example that $B = 45°$, so that $\tan B = 1$; then we shall obtain
$$\tan(A+45°) = \frac{1+\tan A}{1-\tan A}; \quad \tan(A-45°) = \frac{\tan A - 1}{\tan A + 1}.$$

86. $\text{Cot}\,(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$

$$=\frac{\dfrac{\cos A}{\sin A}\,\dfrac{\cos B}{\sin B}-1}{\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}.$$

Suppose $B=A$; thus we obtain

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A}.$$

Similarly $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

87. $\text{Sin}\,2A=2\sin A\cos A=\dfrac{2\sin A\cos A}{\sin^2 A+\cos^2 A}$ (Arts. 82 and 32); divide both numerator and denominator of the last expression by $\cos^2 A$; thus we get $$\frac{\dfrac{2\sin A}{\cos A}}{1+\dfrac{\sin^2 A}{\cos^2 A}};$$

therefore $$\sin 2A=\frac{2\tan A}{1+\tan^2 A}.$$

Also $\cos 2A=\cos^2 A-\sin^2 A=\dfrac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$ (Arts. 82 and 32)

$$=\frac{1-\dfrac{\sin^2 A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

88. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}$ (Art. 83)

$$=\frac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}};$$

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}} \quad \text{(Art. 83)}$$

$$= \cot\frac{A+B}{2}\cot\frac{A-B}{2}.$$

89. $\text{Tan } A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$

$$= \frac{\sin(A+B)}{\cos A \cos B}.$$

Similarly $\tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}.$

90. $\text{Tan } A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$

$$= \frac{1}{\sin A \cos A} = \frac{2}{2\sin A \cos A} = \frac{2}{\sin 2A}.$$

$$\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\sin A \cos A}$$

$$= -\frac{\cos 2A}{\sin A \cos A} = -\frac{2\cos 2A}{\sin 2A} = -2\cot 2A.$$

91. $\text{Sin } 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$$

$$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A$$

$$= 3\sin A - 4\sin^3 A.$$

$$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$$

$$= (2\cos^2 A - 1)\cos A - 2\cos A \sin^2 A$$

$$= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A)$$

$$= 4\cos^3 A - 3\cos A.$$

Hence $\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}.$

Divide both numerator and denominator by $\cos^3 A$; thus

$$\tan 3A = \frac{\dfrac{3\tan A}{\cos^2 A} - 4\tan^3 A}{4 - \dfrac{3}{\cos^2 A}}$$

$$= \frac{3\tan A(1+\tan^2 A) - 4\tan^3 A}{4 - 3(1+\tan^2 A)} \text{ (Art. 34)} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

92. *To find the values of the Trigonometrical Ratios for an angle of* $15°$ *and an angle of* $75°$.

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{\sqrt{3}-1}{2\sqrt{2}};$$

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{3}+1}{2\sqrt{2}};$$

$$\tan 15° = \frac{\sin 15°}{\cos 15°} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{2} = 2 - \sqrt{3};$$

$$\cot 15° = \frac{\cos 15°}{\sin 15°} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2} = 2 + \sqrt{3};$$

$$\sec 15° = \frac{1}{\cos 15°} = \frac{2\sqrt{2}}{\sqrt{3}+1}; \quad \operatorname{cosec} 15° = \frac{1}{\sin 15°} = \frac{2\sqrt{2}}{\sqrt{3}-1}.$$

And $\sin 75° = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\quad \cos 75° = \sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$;

$\tan 75° = \cot 15° = 2 + \sqrt{3}$; $\quad \cot 75° = \tan 15° = 2 - \sqrt{3}$;

$\sec 75° = \operatorname{cosec} 15° = \dfrac{2\sqrt{2}}{\sqrt{3}-1}$; $\quad \operatorname{cosec} 75° = \sec 15° = \dfrac{2\sqrt{2}}{\sqrt{3}+1}$.

93. *If* $\sin A = \sin B$ *and* $\cos A = \cos B$, *then either* A *and* B *are equal, or they differ by some multiple of four right angles.*

For $\quad \cos(A - B) = \cos A \cos B + \sin A \sin B$

$$= \cos^2 A + \sin^2 A = 1;$$

therefore $A - B = 0$, or a multiple of four right angles taken positively or negatively. (Art. 67.)

94. *If* $\cos A = \cos B$ *and* $\sin A = -\sin B$, *then* A + B is *zero, or a multiple of four right angles positive or negative.*

For the given relations may be written

$$\cos A = \cos(-B), \quad \sin A = \sin(-B). \quad \text{(Art. 49.)}$$

Hence by the preceding article $A - (-B)$, that is $A + B$, is zero or some multiple of four right angles taken positively or negatively.

EXAMPLES.

Prove the following identities:

1. $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A.$

2. $2\sin^2 A \sin^2 B + 2\cos^2 A \cos^2 B = 1 + \cos 2A \cos 2B.$

3. $\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A.$

4. $\sin 3A \operatorname{cosec} A - \cos 3A \sec A = 2.$

5. $3\sin A - \sin 3A = 2\sin A(1 - \cos 2A).$

6. $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}.$

7. $\dfrac{\sin B}{\sin A} = \dfrac{\sin(2A + B)}{\sin A} - 2\cos(A + B).$

8. $\sin 4A = 4\sin A \cos^3 A - 4\cos A \sin^3 A.$

9. $\dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A.$

10. $\dfrac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \tan 3A.$

11. $\operatorname{cosec} 2A + \cot 4A = \cot A - \operatorname{cosec} 4A.$

12. $\cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A \cos B = \sin^2 A.$

13. $\sin^2(A - B) + \sin^2 B + 2\sin(A - B)\sin B \cos A = \sin^2 A.$

14. $\dfrac{1 - \tan^2(45^\circ - A)}{1 + \tan^2(45^\circ - A)} = \sin 2A.$

15. $\dfrac{4\tan A(1 - \tan^2 A)}{(1 + \tan^2 A)^2} = \sin 4A.$

16. $\sin A\,(1+\tan A)+\cos A\,(1+\cot A)=\sec A+\operatorname{cosec} A.$

17. $\dfrac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A}=\dfrac{1+2\sin 2A}{1-2\sin 2A}\tan(A-45^\circ).$

18. $\cos A+\cos(120^\circ-A)+\cos(120^\circ+A)=0.$

19. $4\sin A\sin(60^\circ-A)\sin(60^\circ+A)=\sin 3A.$

20. $4\cos A\cos(120^\circ-A)\cos(120^\circ+A)=\cos 3A.$

21. $\sin 3A\sin^3 A+\cos 3A\cos^3 A=\cos^3 2A.$

22. $\cos^3 A\,\dfrac{\sin 3A}{3}+\sin^3 A\,\dfrac{\cos 3A}{3}=\dfrac{\sin 4A}{4}.$

23. $\cos nA\cos(n+2)A-\cos^2(n+1)A+\sin^2 A=0.$

24. $\dfrac{\sin A+\sin nA+\sin(2n-1)A}{\cos A+\cos nA+\cos(2n-1)A}=\tan nA.$

25. $\sin nA\operatorname{cosec}^2 A\sec A-\cos nA\sec^2 A\operatorname{cosec} A$

$=4\sin(n-1)A\operatorname{cosec}^2 2A.$

26. $\cos 10A+\cos 8A+3\cos 4A+3\cos 2A=8\cos A\cos^3 3A.$

27. $\cot A+\cot 2A+\cot 4A$

$=\operatorname{cosec} 4A\,(2+2\cos 2A+3\cos 4A).$

28. $\operatorname{cosec} A=\dfrac{2\sin 2A+2\cos 2A}{\cos A-\sin A-\cos 3A+\sin 3A}.$

29. $\cos^2 2A=(\cos A-\sin 3A)^2+2\cos A\sin 3A(\cos A-\sin A)^2.$

30. $\cos^6 A-\sin^6 A=\cos 2A\,(1-\tfrac{1}{4}\sin^2 2A).$

Solve the following equations:

31. $\tan\left(\dfrac{\pi}{4}-\theta\right)+\cot\left(\dfrac{\pi}{4}-\theta\right)=4.$

32. $\sin 4\theta+\sin\theta=0.$

33. $\sin 7\theta-\sin\theta=\sin 3\theta.$

34. $\sin\theta+\cos\theta=\dfrac{1}{\sqrt{2}}.$

35. $\sin 5\theta=16\sin^5\theta.$

36. $\cos 3\theta+\cos 2\theta+\cos\theta=0.$

37. $\sin 3\theta+\sin 2\theta+\sin\theta=0.$

38. $\tan\theta+\tan\left(\dfrac{\pi}{4}+\theta\right)=2.$

39. $\tan 2\theta=8\cos^2\theta-\cot\theta.$

40. $\tan\left(\dfrac{\pi}{4}+\theta\right)=3\tan\left(\dfrac{\pi}{4}-\theta\right).$

VII. FORMULÆ FOR THE DIVISION OF ANGLES.

95. In Art. 82 change A into $\frac{A}{2}$; thus we obtain

$$\cos A = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1;$$

$$\text{therefore } \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}, \quad \cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}.$$

96. Since we may suppose either the positive or negative sign to be placed before the radical quantities in the preceding article, we see that corresponding to *one* value of $\cos A$ there are *two* values of $\sin\frac{A}{2}$ and *two* values of $\cos\frac{A}{2}$; and the reason of this may be assigned. For if α be an angle which has a given cosine, then the formula $2n\pi \pm \alpha$ includes all the angles which have this given cosine; therefore any expression which gives the value of $\sin\frac{\alpha}{2}$ in terms of $\cos\alpha$ may be expected to give the value of the sine of every angle included in the formula $\frac{1}{2}(2n\pi \pm \alpha)$. Now

$$\sin\left(n\pi \pm \frac{\alpha}{2}\right) = \sin n\pi \cos\frac{\alpha}{2} \pm \cos n\pi \sin\frac{\alpha}{2}$$

$$= \pm \cos n\pi \sin\frac{\alpha}{2} = \pm \sin\frac{\alpha}{2};$$

thus two values occur which differ only in sign. Similarly, any expression which gives the value of $\cos\frac{\alpha}{2}$ in terms of $\cos\alpha$ may be expected to give the value of the cosine of every angle included in the formula $\frac{1}{2}(2n\pi \pm \alpha)$. Now

$$\cos\left(n\pi \pm \frac{\alpha}{2}\right) = \cos n\pi \cos\frac{\alpha}{2} \mp \sin n\pi \sin\frac{\alpha}{2}$$

$$= \cos n\pi \cos\frac{\alpha}{2} = \pm \cos\frac{\alpha}{2};$$

thus two values occur which differ only in sign.

97. If $\cos A$ only be given and nothing more be known respecting A, then the ambiguity of sign which occurs in Art. 95 cannot be removed. If however A itself be given, then $\frac{A}{2}$ is a known angle, and therefore we know whether $\sin\frac{A}{2}$ is positive or negative; and also whether $\cos\frac{A}{2}$ is positive or negative; thus we know which sign is to be taken with each radical quantity. Or if we merely know in which quadrant the angle $\frac{A}{2}$ lies, we can determine the proper signs; for example, if $\frac{A}{2}$ is an angle between 180° and 270°, both its sine and cosine must be negative quantities.

98. By Art. 82 $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$,

also $1 = \sin^2\frac{A}{2} + \cos^2\frac{A}{2}$,

thus $\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2 = 1 + \sin A$,

and $\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 = 1 - \sin A$;

therefore $\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{(1 + \sin A)}$..................(1),

and $\sin\frac{A}{2} - \cos\frac{A}{2} = \sqrt{(1 - \sin A)}$..................(2);

therefore $2\sin\frac{A}{2} = \sqrt{(1 + \sin A)} + \sqrt{(1 - \sin A)}$,

and $2\cos\frac{A}{2} = \sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)}$.

99. Since we may suppose either the positive or negative sign to be placed before each of the radical quantities in equations (1) and (2) of the preceding article, we see that corresponding to *one* value of $\sin A$ there are *four* values for $\cos\frac{A}{2}$ and *four* values for

$\sin\frac{A}{2}$, and the reason of this may be assigned. For if a be an angle which has a given sine, then the formula $n\pi+(-1)^n a$ includes all the angles which have this given sine; therefore any expression which gives the value of $\sin\frac{a}{2}$ in terms of $\sin a$ may be expected to give the value of the sine of every angle included in the formula $\frac{1}{2}\{n\pi+(-1)^n a\}$. First suppose n even and equal to $2m$; then

$$\sin\tfrac{1}{2}\{n\pi+(-1)^n a\}=\sin\left(m\pi+\frac{a}{2}\right)=\sin m\pi\cos\frac{a}{2}+\cos m\pi\sin\frac{a}{2}$$

$$=\cos m\pi\sin\frac{a}{2}=\pm\sin\frac{a}{2}.$$

Next suppose n odd and equal to $2m+1$; then

$$\sin\tfrac{1}{2}\{n\pi+(-1)^n a\}=\sin\left(m\pi+\frac{\pi-a}{2}\right)=\sin m\pi\cos\frac{\pi-a}{2}+\cos m\pi\sin\frac{\pi-a}{2}$$

$$=\cos m\pi\sin\frac{\pi-a}{2}=\pm\sin\frac{\pi-a}{2}=\pm\cos\frac{a}{2}.$$

Thus four values occur for the sine of half an angle when the sine of the angle is given.

Similarly any expression which gives the value of $\cos\frac{a}{2}$ in terms of $\sin a$, may be expected to give the value of the cosine of every angle included in the formula $\frac{1}{2}\{n\pi+(-1)^n a\}$. First suppose n even and equal to $2m$; then

$$\cos\tfrac{1}{2}\{n\pi+(-1)^n a\}=\cos\left(m\pi+\frac{a}{2}\right)=\cos m\pi\cos\frac{a}{2}-\sin m\pi\sin\frac{a}{2}$$

$$=\cos m\pi\cos\frac{a}{2}=\pm\cos\frac{a}{2}.$$

Next suppose n odd and equal to $2m+1$; then

$$\cos\tfrac{1}{2}\{n\pi+(-1)^n a\}=\cos\left(m\pi+\frac{\pi-a}{2}\right)=\cos m\pi\cos\frac{\pi-a}{2}-\sin m\pi\sin\frac{\pi-a}{2}$$

$$=\cos m\pi\cos\frac{\pi-a}{2}=\pm\cos\frac{\pi-a}{2}=\pm\sin\frac{a}{2}.$$

Thus four values occur for the cosine of half an angle when the sine of the angle is given.

100. If $\sin A$ only be given and nothing more be known respecting A, then the ambiguities of sign which occur in Art. 98 cannot be removed. If however A itself be given, or if we merely know in which quadrant the angle A lies, we can determine the proper signs; for in any particular case we may proceed as follows. We have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{(1+\sin A)} \quad \text{.....................(1)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{(1-\sin A)} \quad \text{.....................(2)}.$$

Now suppose, for example, that A lies between 0 and 90°, then $\frac{A}{2}$ lies between 0 and 45°; therefore $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$ are both positive and $\cos\frac{A}{2}$ is greater than $\sin\frac{A}{2}$; hence the left-hand member of (1) is a *positive* quantity, and we must therefore take the *positive* sign in (1), and the left-hand member of (2) is a *negative* quantity, and we must therefore take the *negative* sign in (2). Therefore if A lies between 0 and 90°, we have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=+\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)};$$

therefore $$2\sin\frac{A}{2}=+\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)},$$

$$2\cos\frac{A}{2}=+\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)}.$$

For another example, suppose that A lies between 270° and 360°, then $\frac{A}{2}$ lies between 135° and 180°; therefore $\cos\frac{A}{2}$ is negative, and $\sin\frac{A}{2}$ is positive, and $\cos\frac{A}{2}$ is numerically greater than $\sin\frac{A}{2}$;

hence the left-hand member of (1) is a *negative* quantity, and we must therefore take the *negative* sign in (1), and the left-hand member of (2) is a *positive* quantity, and we must therefore take the *positive* sign in (2). Therefore if A lies between 270° and 360°, we have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=+\sqrt{(1-\sin A)};$$

therefore $$2\sin\frac{A}{2}=-\sqrt{(1+\sin A)}+\sqrt{(1-\sin A)},$$

$$2\cos\frac{A}{2}=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

101. It is easy to give general formulæ for determining the signs of $\sin\frac{A}{2}+\cos\frac{A}{2}$ and $\sin\frac{A}{2}-\cos\frac{A}{2}$. For

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)=\sqrt{2}\sin\left(\frac{A}{2}+\frac{\pi}{4}\right);$$

now $\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)$ is *positive* if $\frac{A}{2}+\frac{\pi}{4}$ lies between $2n\pi$ and $(2n+1)\pi$, and *negative* if $\frac{A}{2}+\frac{\pi}{4}$ lies between $(2n+1)\pi$ and $(2n+2)\pi$, where n is zero or any integer positive or negative. Thus $\sin\frac{A}{2}+\cos\frac{A}{2}$ is *positive* if $\frac{A}{2}$ lies between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$, and negative if $\frac{A}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$. Similarly

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right);$$

and hence we can infer that $\sin\frac{A}{2}-\cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lies between $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$, and negative if $\frac{A}{2}$ lies between $2n\pi+\frac{5\pi}{4}$ and $2n\pi+\frac{9\pi}{4}$.

102. By Art. 85, $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$;

put c for $\tan A$; thus $c\tan^2\frac{A}{2} + 2\tan\frac{A}{2} - c = 0$;

therefore $\tan\frac{A}{2} = \dfrac{-1 \pm \sqrt{(1+c^2)}}{c}$.

103. The reason why two values occur in finding the tangent of half an angle when the tangent of the angle is given, may be assigned as before. For if α be an angle which has a given tangent, then the formula $n\pi + \alpha$ includes all the angles which have this given tangent; therefore any expression which gives the value of $\tan\frac{\alpha}{2}$ in terms of $\tan\alpha$ may be expected to give the value of the tangent of every angle included in the formula $\frac{1}{2}(n\pi + \alpha)$. First suppose n even and equal to $2m$; then

$$\tan\tfrac{1}{2}(n\pi + \alpha) = \tan\left(m\pi + \frac{\alpha}{2}\right) = \tan\frac{\alpha}{2}.$$

Next suppose n odd and equal to $2m+1$, then

$$\tan\tfrac{1}{2}(n\pi + \alpha) = \tan\left(m\pi + \frac{\pi+\alpha}{2}\right) = \tan\frac{\pi+\alpha}{2} = \tan\left(\frac{\pi}{2} + \frac{\alpha}{2}\right) = -\cot\frac{\alpha}{2}.$$

Thus two values occur for the tangent of half an angle when the tangent of the angle is given.

104. If $\tan A$ only be given and nothing more be known respecting A, then the ambiguity of sign which occurs in Art. 102 cannot be removed. If however A itself be given, or if we merely know in which quadrant $\frac{A}{2}$ lies, we know whether $\tan\frac{A}{2}$ is positive or negative, and thus we know which sign we must take.

105. By Art. 91, $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$.

Thus if $\cos A$ be given we have a *cubic* equation for determining

$\cos\frac{A}{3}$; and the reason for this may be assigned as before. For if α be an angle which has a given cosine, then the formula $2n\pi \pm \alpha$ includes all the angles which have this given cosine; therefore any expression which gives the value of $\cos\frac{\alpha}{3}$ in terms of $\cos\alpha$ may be expected to give the value of the cosine of every angle included in the formula $\frac{1}{3}(2n\pi \pm \alpha)$. Now n is of one of the forms $3m$, $3m+1$, $3m-1$. First suppose $n = 3m$; then

$$\cos\tfrac{1}{3}(2n\pi \pm \alpha) = \cos\left(2m\pi \pm \frac{\alpha}{3}\right) = \cos\frac{\alpha}{3}.$$

Next suppose $n = 3m+1$; then

$$\cos\tfrac{1}{3}(2n\pi \pm \alpha) = \cos\left(2m\pi + \frac{2\pi \pm \alpha}{3}\right) = \cos\frac{2\pi \pm \alpha}{3}.$$

Last suppose $n = 3m-1$; then

$$\cos\tfrac{1}{3}(2n\pi \pm \alpha) = \cos\left(2m\pi - \frac{2\pi \pm \alpha}{3}\right) = \cos\frac{2\pi \mp \alpha}{3}.$$

Thus three values occur, namely $\cos\frac{\alpha}{3}$, $\cos\frac{2\pi+\alpha}{3}$, $\cos\frac{2\pi-\alpha}{3}$.

106. By Art. 91, $\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$.

Thus if $\sin A$ be given, we have a *cubic* equation for determining $\sin\frac{A}{3}$; and the reason for this may be assigned as before.

EXAMPLES.

1. Shew that $2\sin\frac{A}{2} = -\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}$, when A lies between 450° and 630°.

2. Obtain $\cos\frac{A}{2}$ in terms of $\sin A$ when $\frac{A}{2}$ lies between 405° and 495°.

3. Obtain $\sin\frac{A}{2}$ in terms of $\sin A$ when $\frac{A}{2}$ lies between $-45°$ and $-135°$.

4. Determine the limits between which A must lie in order that

$$2\sin A = -\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)},$$

and
$$2\cos A = -\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}.$$

5. Determine the limits between which A must lie in order that

$$2\cos A = -\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}.$$

6. Determine the limits between which A must lie in order that

$$2\sin A = \sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}.$$

7. Divide a given angle into two parts whose sines shall be in a given ratio.

8. Divide a given angle into two parts whose cosines shall be in a given ratio.

9. Divide a given angle into two parts whose tangents shall be in a given ratio.

10. Given $\tan\frac{A}{2} = 2 - \sqrt{3}$, find $\sin A$.

11. Given $\sin 210° = -\frac{1}{2}$, find $\cos 105°$.

12. Given $\tan 2A = -\frac{24}{7}$, find $\sin A$ and $\cos A$.

13. Find $\tan 165°$ from the known value of $\tan 330°$.

14. Shew that $\tan^2\frac{A}{2} = \frac{2\sin A - \sin 2A}{2\sin A + \sin 2A}$.

15. $\text{vers}\,(180° - A) = 2\,\text{vers}\,\frac{1}{2}(180° + A)\,\text{vers}\,\frac{1}{2}(180° - A)$.

16. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\frac{1}{2}(A - B)$.

17. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4 \sin^2 \frac{1}{2}(A - B)$.

18. Shew that $\sin 22\frac{1}{2}^\circ = \frac{\sqrt{(2-\sqrt{2})}}{2}$, $\cos 22\frac{1}{2}^\circ = \frac{\sqrt{(2+\sqrt{2})}}{2}$,

and $$\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1.$$

19. $(\tan A + \cot A)\, 2 \tan \frac{A}{2}\left(1 - \tan^2 \frac{A}{2}\right) = \left(1 + \tan^2 \frac{A}{2}\right)^2$.

20. $\tan^2\left(\frac{\pi}{4} + \frac{A}{2}\right) = \frac{\sec A + \tan A}{\sec A - \tan A}$.

21. $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{\sin \theta}{\sqrt{(\text{vers}\, \theta)}}$.

22. $\sqrt{(1 + \sin \theta)} = 1 + 2 \sin \frac{\theta}{4} \sqrt{\left(1 - \sin \frac{\theta}{2}\right)}$.

23. $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$.

24. $\tan 7\frac{1}{2}^\circ = \frac{\sqrt{2} - 1}{\sqrt{2} + \sqrt{3}}$.

25. $\tan 142\frac{1}{2}^\circ = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

26. If $\tan x = (2 + \sqrt{3}) \tan \frac{x}{3}$, find the value of $\tan x$.

27. If $\alpha = \left(n + \frac{1}{4} \pm \frac{1}{6}\right)\pi$, find $\tan \alpha + \cot \alpha$.

28. If $\alpha = \frac{\pi}{17}$, find the value of $\frac{\cos \alpha \cos 13\alpha}{\cos 3\alpha + \cos 5\alpha}$.

29. If $\sec(\phi + \alpha) + \sec(\phi - \alpha) = 2 \sec \phi$, shew that

$$\cos \phi = \sqrt{2} \cos \frac{\alpha}{2}.$$

30. If $\tan \frac{\theta}{2} = \left(\frac{1+c}{1-c}\right)^{\frac{1}{2}} \tan \frac{\phi}{2}$, shew that

$$\cos \theta = \frac{\cos \phi - c}{1 - c \cos \phi}.$$

VIII. MISCELLANEOUS PROPOSITIONS.

107. *To find the sine and cosine of an angle of* 18°.

Let A denote an angle which contains 18°, then $2A$ contains 36° and $3A$ contains 54°; hence $\sin 2A = \cos 3A$,

therefore $$2 \sin A \cos A = 4 \cos^3 A - 3 \cos A;$$

divide by $\cos A$, thus $2 \sin A = 4 \cos^2 A - 3 = 1 - 4 \sin^2 A$,

therefore $$4 \sin^2 A + 2 \sin A - 1 = 0;$$

by solving this quadratic equation we obtain

$$\sin A = \frac{-1 \pm \sqrt{5}}{4}.$$

Since the sine of an angle of 18° is a *positive* quantity we must take the upper sign, therefore

$$\sin 18° = \frac{\sqrt{5} - 1}{4},$$

and $$\cos 18° = \sqrt{(1 - \sin^2 18°)} = \frac{\sqrt{(10 + 2\sqrt{5})}}{4}.$$

108. *To find the sine and cosine of an angle of* 36°.

$$\cos 36° = 1 - 2 \sin^2 18° = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2 = \frac{1 + \sqrt{5}}{4},$$

$$\sin 36° = \sqrt{(1 - \cos^2 36°)} = \frac{\sqrt{(10 - 2\sqrt{5})}}{4}.$$

109. Hence the values of the Trigonometrical Ratios for angles of 54° and 72° are known; for

$\sin 54° = \cos 36°$, $\cos 54° = \sin 36°$, $\sin 72° = \cos 18°$, $\cos 72° = \sin 18°$.

110. The reason why more than one result was obtained in Art. 107, is that the equation $\sin 2A = \cos 3A$ is true for some other angles besides the angle which contains 18°. This equation may be written

$$\cos (90° - 2A) = \cos 3A,$$

Hence we conclude that $90° - 2A$ must either be equal to $3A$ or to one of the angles which have the same cosine as $3A$; thus every admissible value of A will be found from the equation

$$90° - 2A = n \,.\, 360° \pm 3A;$$

where n is zero or any integer positive or negative;

thus $$A = \frac{90° - n \,.\, 360°}{2 \pm 3}.$$

For example, if $n = 0$ and we take the lower sign in the denominator, we obtain $A = -90°$; this value of A makes $\cos A = 0$, and thus we see a reason for the appearance of the factor $\cos A$ which was removed by division in Art. 107. Again, if we put $n = 1$ and take the upper sign in the denominator, we obtain $A = -\frac{270°}{5} = -54°$; and $\sin(-54°) = -\sin 54° = -\cos 36° = -\frac{1+\sqrt{5}}{4}$; and thus we see a reason for the appearance of the other root in the quadratic equation of Art. 107, besides the root which we used.

111. *To find the sine and cosine of an angle of* 9°, *and of an angle of* 81°.

By Art. 100,

$$\sin 9° + \cos 9° = \sqrt{(1 + \sin 18°)} = \frac{\sqrt{(3+\sqrt{5})}}{2},$$

$$\sin 9° - \cos 9° = -\sqrt{(1 - \sin 18°)} = -\frac{\sqrt{(5-\sqrt{5})}}{2};$$

therefore $$\sin 9° = \frac{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}}{4},$$

$$\cos 9° = \frac{\sqrt{(3+\sqrt{5})} + \sqrt{(5-\sqrt{5})}}{4}.$$

And $$\sin 81° = \cos 9°, \quad \cos 81° = \sin 9°.$$

We have now found expressions for the sines and cosines of the following angles, 9°, 15°, 18°, 30°, 36°, 45°, 54°, 60°, 72°, 75°, 81°. (See Arts. 36, 37, 92, 107, 108, 111.)

Since $3° = 18° - 15°$, we can obtain the sine and cosine of 3° from those of 18° and 15° by Art. 77; and then by means of Art. 76 combined with results already obtained, we can easily find the sines and cosines of any angle comprised in the series 3°, 6°, 9°, 12°, &c.

112. In Arts. 87 and 91 we have given expressions for $\sin 2A$, $\cos 2A$, $\sin 3A$, and $\cos 3A$ in terms of $\sin A$ and $\cos A$; we may also express the sines and cosines of $4A$, $5A$, &c. in a similar way.

For $\sin(n+1)A + \sin(n-1)A = 2\sin nA \cos A$;

therefore $\quad \sin(n+1)A = 2\sin nA\cos A - \sin(n-1)A$;

let $\quad n=3$; thus $\sin 4A = 2\sin 3A\cos A - \sin 2A$;

let $\quad n=4$; thus $\sin 5A = 2\sin 4A\cos A - \sin 3A$;

and so on; thus we can find in succession $\sin 4A$, $\sin 5A$, &c., in terms of the sine and cosine of A.

Similarly, the formula

$$\cos(n+1)A + \cos(n-1)A = 2\cos nA\cos A,$$

may be used to find in succession $\cos 4A$, $\cos 5A$, &c.

This subject will be considered again hereafter, and we shall then give general formulæ for the sine and cosine of nA in terms of the sine and cosine of A for any integral value of n.

113. It is easy to find expressions for the Trigonometrical Ratios of any compound angle in terms of the Ratios of the component angles. For example,

$$\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= \sin A\cos B\cos C + \sin B\cos C\cos A$$
$$+ \sin C\cos A\cos B - \sin A\sin B\sin C.$$

$$\text{Cos}(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= \cos A\cos B\cos C - \cos A\sin B\sin C$$
$$- \cos B\sin A\sin C - \cos C\sin A\sin B.$$

$$\text{Tan}(A+B+C) = \frac{\sin(A+B+C)}{\cos(A+B+C)}$$

$$= \frac{\sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C}{\cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin A\sin C - \cos C\sin A\sin B};$$

divide both numerator and denominator of the last expression by

$\cos A \cos B \cos C$; thus we obtain

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

Suppose B and C each equal to A; thus we have

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

114. When three or more angles are connected by some relation, we may often find that some simple relation exists among some of their Trigonometrical Ratios, thus, for example,

if $A + B + C = 180°$, then will

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

For $\quad \sin 2A + \sin 2B = 2 \sin(A+B) \cos(A-B) = 2 \sin C \cos(A-B)$

and $\quad \sin 2C = 2 \sin C \cos C = -2 \sin C \cos(A+B)$, (Art. 48);

therefore

$$\sin 2A + \sin 2B + \sin 2C = 2 \sin C \{\cos(A-B) - \cos(A+B)\}$$
$$= 4 \sin C \sin A \sin B.$$

Again, if $A + B + C = 180°$, then will

$$\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$$

For $\quad \cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$

$$= 2 \sin \tfrac{1}{2} C \cos \tfrac{1}{2}(A-B);$$

and $\quad \cos C = 1 - 2 \sin^2 \frac{1}{2} C$; therefore

$$\cos A + \cos B + \cos C = 1 + 2 \sin \tfrac{1}{2} C \left\{ \cos \tfrac{1}{2}(A-B) - \sin \tfrac{1}{2} C \right\}$$
$$= 1 + 2 \sin \tfrac{1}{2} C \left\{ \cos \tfrac{1}{2}(A-B) - \cos \tfrac{1}{2}(A+B) \right\}$$
$$= 1 + 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$$

Again, if $A + B + C = 180°$, then will

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

For $\tan 180° = 0$, therefore $\tan(A+B+C)=0$; and therefore by Art. 113, $\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$.

Again, by Art. 113,

$$\cot(A+B+C) = \frac{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}{\tan A + \tan B + \tan C - \tan A \tan B \tan C};$$

now $\cot 90° = 0$; hence if $A+B+C = 90°$, then will

$$1 = \tan B \tan C + \tan C \tan A + \tan A \tan B.$$

115. For another example, suppose we have to investigate what relation must exist among the angles A, B, C, in order that

$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C - 1$ may be zero.

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C - 1$$
$$= (\cos A + \cos B \cos C)^2 + \cos^2 B + \cos^2 C - 1 - \cos^2 B \cos^2 C$$
$$= (\cos A + \cos B \cos C)^2 + 1 - \sin^2 B + 1 - \sin^2 C - 1$$
$$- (1 - \sin^2 B)(1 - \sin^2 C)$$
$$= (\cos A + \cos B \cos C)^2 - \sin^2 B \sin^2 C$$
$$= (\cos A + \cos B \cos C + \sin B \sin C)(\cos A + \cos B \cos C - \sin B \sin C)$$
$$= \{\cos A + \cos(B-C)\}\{\cos A + \cos(B+C)\}$$
$$= 4\cos\frac{A+B-C}{2}\cos\frac{A-B+C}{2}\cos\frac{A+B+C}{2}\cos\frac{B+C-A}{2}.$$

Hence in order that the proposed expression may be zero, one of the four cosines last written must be zero, and thus one of the four compound angles must be some odd multiple of a right angle.

MISCELLANEOUS EXAMPLES.

Prove the following formulæ:

1. $\dfrac{\cos(A+B+C)}{\cos A \cos B \cos C} = 1 - \tan B \tan C - \tan C \tan A - \tan A \tan B.$

2. $\dfrac{\sin(A+B+C)}{\cos A \cos B \cos C} = \tan A + \tan B + \tan C - \tan A \tan B \tan C.$

3. $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$
$$+4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}=0.$$

4. $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$
$=1+\cos(2\theta-2m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).$

5. $\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma=\sin(\beta-\gamma)\cos(\alpha+\beta+\gamma).$

6. $\cos(A+B+C)+\cos(A+B-C)+\cos(A+C-B)$
$+\cos(B+C-A)=4\cos A\cos B\cos C.$

7. $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha)$
$-\cos 2(\alpha+\beta+\gamma).$

8. $$\frac{\sin A}{\sin(A-B)\sin(A-C)}+\frac{\sin B}{\sin(B-C)\sin(B-A)}$$
$$+\frac{\sin C}{\sin(C-A)\sin(C-B)}=0.$$

9. $\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\gamma)\sin\gamma$
$=\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma.$

10. $\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$
$=\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta-\gamma)\}.$

11. $\sin(A+B+C)\sin B=\sin(A+B)\sin(B+C)-\sin A\sin C.$

12. $\sin\alpha\sin\beta\sin(\beta-\alpha)+\sin\beta\sin\gamma\sin(\gamma-\beta)$
$+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)=0.$

13. $\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)$
$+\cos(\gamma+\delta)\sin(\gamma-\delta)+\cos(\delta+\alpha)\sin(\delta-\alpha)=0.$

14. $\sin(\delta-\beta)\sin(\alpha-\gamma)+\sin(\beta-\gamma)\sin(\alpha-\delta)$
$+\sin(\gamma-\delta)\sin(\alpha-\beta)=0.$

If $A+B+C=180°$, prove the following formulæ contained in the examples from 15 to 35 inclusive.

15. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

16. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

17. $\sin A - \sin B + \sin C = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$.

18. $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0$.

19. $\cos 4A + \cos 4B + \cos 4C + 1 = 4 \cos 2A \cos 2B \cos 2C$.

20. $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$.

21. $\cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi + A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi + C}{4}$.

22. $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1 = 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}$.

23. $\sin^2 A + \sin^2 B + \sin^2 C - 2 \cos A \cos B \cos C = 2$.

24. $\sin^2 2A + \sin^2 2B + \sin^2 2C + 2 \cos 2A \cos 2B \cos 2C = 2$.

25. $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$.

26. $\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2}$.

27. $1 + \cos A \cos B \cos C = \cos A \sin B \sin C + \cos B \sin A \sin C$
$+ \cos C \sin A \sin B$.

28. $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
$+ \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

29. $\sin^2 \frac{C}{2} = \frac{(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)}{4 \sin A \sin B}$.

30. The expression $\cot A + \frac{\sin A}{\sin B \sin C}$ will retain the same value if any two of the quantities A, B, C, be interchanged.

31. $\frac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}$.

32. $\sin nA + \sin nB + \sin nC = 4 \sin \frac{n\pi}{2} \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2}$, if n be an integer of the form $4m + 1$ or $4m + 3$.

33. $\sin nA + \sin nB + \sin nC = -4\cos\frac{n\pi}{2}\sin\frac{nA}{2}\sin\frac{nB}{2}\sin\frac{nC}{2}$, if n be an integer of the form $4m$ or $4m+2$.

34. $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{B+C}{4}\cos\frac{A+C}{4}\cos\frac{A+B}{4}$.

35. $\frac{\tan A}{\tan B} + \frac{\tan B}{\tan C} + \frac{\tan C}{\tan A} + \frac{\tan A}{\tan C} + \frac{\tan B}{\tan A} + \frac{\tan C}{\tan B}$

$$= \sec A \sec B \sec C - 2.$$

36. If the sum of four angles be two right angles, the sum of their tangents is equal to the sum of the product of the tangents taken three and three.

37. If $\frac{\tan(A-B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, prove that $\tan A \tan B = \tan^2 C$.

38. Given $\frac{\tan^2\alpha}{\tan^2\beta} = \frac{\cos\beta(\cos x - \cos\alpha)}{\cos\alpha(\cos x - \cos\beta)}$,

shew that $$\tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}.$$

39. If $\cos^2\theta = \frac{\cos\alpha}{\cos\beta}$, $\cos^2\theta' = \frac{\cos\alpha'}{\cos\beta}$, $\frac{\tan\theta}{\tan\theta'} = \frac{\tan\alpha}{\tan\alpha'}$,

shew that $$\tan^2\frac{\alpha}{2}\tan^2\frac{\alpha'}{2} = \tan^2\frac{\beta}{2}.$$

40. If $\cos\alpha = \cos\beta\cos\phi = \cos\beta'\cos\phi'$, and

$\sin\alpha = 2\sin\frac{\phi}{2}\sin\frac{\phi'}{2}$, shew that $\tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}\tan^2\frac{\beta'}{2}$.

41. If $\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\alpha+\theta)}{\sin\theta}$, shew that

$$\cot\beta - \cot\theta = \cot(\alpha+\theta) + \cot(\alpha-\beta).$$

42. If $\left(\frac{\tan\alpha}{\sin\theta} - \frac{\tan\beta}{\tan\theta}\right)^2 = \tan^2\alpha - \tan^2\beta$, then $\cos\theta = \frac{\tan\beta}{\tan\alpha}$.

43. If $\tan\phi = \cos\theta\tan\alpha$, and $\tan\alpha' = \tan\theta\sin\phi$,

then one value of $\tan^2\frac{\phi}{2}$ is $\tan\frac{\alpha+\alpha'}{2}\tan\frac{\alpha-\alpha'}{2}$.

44. Find the relation between the angles α, β, γ, when the cosines are connected by the relation

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=0.$$

45. If $\dfrac{\tan(\theta+\alpha)}{x}=\dfrac{\tan(\theta+\beta)}{y}=\dfrac{\tan(\theta+\gamma)}{z}$, then will

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta)+\frac{y+z}{y-z}\sin^2(\beta-\gamma)+\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=0.$$

46. If $\dfrac{\tan^2\theta}{\tan^2\alpha}+\dfrac{\tan^2\phi}{\tan^2\beta}=1$, and $\dfrac{\sin\theta}{\sin\alpha}=\dfrac{\sin\phi}{\sin\beta}$,

shew that $$\sin\theta=\frac{\pm\sin\alpha}{\sqrt{(1\pm\cos\alpha\cos\beta)}}.$$

47. If $\dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\dfrac{a}{b}$ and $\dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\dfrac{a'}{b'}$,

then $$\cos(\alpha-\beta)=\frac{aa'+bb'}{ab'+a'b}.$$

48. Having given $\tan\phi=\dfrac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta}$, shew that one of the values of $\tan\dfrac{\phi}{2}$ is $\tan\dfrac{\theta}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\theta'}{2}\right)$.

49. Given $\cos\theta=\cos\alpha\cos\beta$, $\cos\theta'=\cos\alpha'\cos\beta$,

$\tan\dfrac{\theta}{2}\tan\dfrac{\theta'}{2}=\tan\dfrac{\beta}{2}$, shew that $\sin^2\beta=(\sec\alpha-1)(\sec\alpha'-1)$.

50. Having given that $\sin(B+C-A)$, $\sin(C+A-B)$, and $\sin(A+B-C)$ are in arithmetical progression, shew that $\tan A$, $\tan B$ and $\tan C$, are in arithmetical progression.

51. If the sines of the angles of a triangle be in arithmetical progression, the cotangents of the half angles are also in arithmetical progression.

52. If the sum of the squares of the cosines of the angles of a triangle $=1$, the difference between the greatest and least angle is equal to the mean angle.

53. If A, B, C be the angles of a triangle, and

$$\sin\left(A+\frac{C}{2}\right)=n\sin\frac{C}{2}, \text{ shew that } \tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}.$$

54. If A, B, C be the angles of a triangle, and

$$\frac{\sin A}{x}=\frac{\sin B}{y}=\frac{\sin C}{z}, \text{ then}$$

$$(x-y)\cot\frac{C}{2}+(y-z)\cot\frac{A}{2}+(z-x)\cot\frac{B}{2}=0.$$

55. If $A+B+C=m\pi$ where m is any integer, then

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

56. If α, β, γ be any angles, shew that

$$\sin\alpha+\sin\beta+\sin\gamma-4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$=2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\left\{\cos\frac{3\alpha-\beta-\gamma+\pi}{4}+\cos\frac{3\beta-\alpha-\gamma+\pi}{4}\right.$$

$$\left.+\cos\frac{3\gamma-\alpha-\beta+\pi}{4}+\cos\frac{\alpha+\beta+\gamma-\pi}{4}\right\}.$$

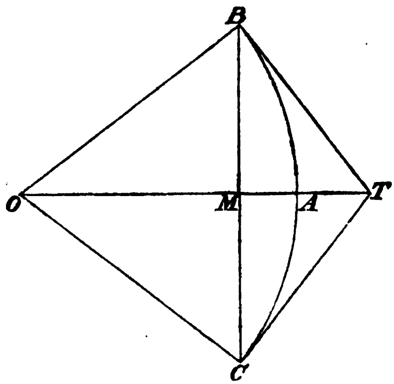
IX. CONSTRUCTION OF TRIGONOMETRICAL TABLES.

116. *If θ be the circular measure of a positive angle less than a right angle, θ is greater than* $\sin\theta$ *and less than* $\tan\theta$.

Let AOB be an angle less than a right angle and let $OB=OA$; from B draw BM perpendicular to OA and produce it to C so that $MC=MB$; draw BT at right angles to OB meeting OA produced in T, and join CT. Then the triangles MOC and MOB are equal in all respects, so that the angle $TOC=$ the angle TOB; therefore the triangles TOC and TOB are equal in all respects, so that TCO is a right angle, and $TC=TB$.

With centre O and radius OB describe an arc of a circle BAC; this will touch BT at B and CT at C.

Now we assume as an axiom that the straight line BC is less than the arc BAC; thus BM the half of BC is less than BA the half of the arc BAC; therefore $\frac{BM}{OB}$ is less than $\frac{BA}{OB}$; that is, the sine of AOB is less than the circular measure of AOB.



Again, we assume as an axiom that the arc BAC is less than the sum of the two exterior lines BT and TC; thus BA is less than BT; therefore $\frac{BA}{OB}$ is less than $\frac{BT}{OB}$; that is, the circular measure of AOB is less than the tangent of AOB.

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude if θ be less than $\frac{\pi}{2}$.

117. We have assumed two axioms in the preceding article; the first is so obvious that it will be readily admitted; but the second is more difficult. The student is recommended to postpone this point for future consideration. It is however not difficult to shew that the assumption may be made to depend upon another almost identical with that which we have already been compelled to make in Art. 14. For divide the arc BAC into any number of arcs and draw tangents at the points of division; then from the fact

that two sides of a triangle are greater than the third, it follows that the perimeter of the portion of a polygon thus formed, is less than the sum of BT and TC by a *finite* difference. Moreover this perimeter diminishes as the number of points of division is increased. Now assume as in Art. 14 that the perimeter of the polygon can be made to *differ as little as we please* from the arc BAC by sufficiently increasing the number of sides and diminishing the length of each side; thus it follows that the arc BAC is less than the sum of BT and TC.

118. *The limit of* $\frac{\sin\theta}{\theta}$ *when* θ *is indefinitely diminished is unity.*

For $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude; divide by $\sin\theta$; therefore 1, $\frac{\theta}{\sin\theta}$, and $\frac{1}{\cos\theta}$ are in ascending order of magnitude. Thus $\frac{\theta}{\sin\theta}$ lies in value between 1 and $\frac{1}{\cos\theta}$; but when θ is zero, $\cos\theta$ is unity; hence as θ diminishes indefinitely $\frac{\theta}{\sin\theta}$ approaches the limit unity. Therefore also $\frac{\sin\theta}{\theta}$ approaches the limit unity.

And as $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}$, the limit of $\frac{\tan\theta}{\theta}$ when θ is indefinitely diminished is also unity.

119. It must be carefully remembered that in the important proposition of the preceding article, θ is the *circular measure* of the angle considered. If any other unit of angular measurement be adopted instead of the unit of circular measure, the limit under consideration will *not* be unity. For example, let us find the limit of $\frac{\sin n^\circ}{n}$ when n is indefinitely diminished. Let θ be the circular measure of an angle of n degrees, then $\theta = \frac{n\pi}{180}$; thus

$$\frac{\sin n^\circ}{n} = \frac{\sin\theta}{\frac{180}{\pi}\theta} = \frac{\pi}{180} \cdot \frac{\sin\theta}{\theta}.$$

Now when n diminishes indefinitely, θ does so also, and the limit of $\frac{\sin\theta}{\theta}$ is unity; hence the limit of $\frac{\sin n^0}{n}$ when n is diminished indefinitely is $\frac{\pi}{180}$, which is the circular measure of an angle of *one degree*. Similarly we may prove that the limit of $\frac{\sin n'}{n}$ when n is indefinitely diminished is the circular measure of an angle of *one minute;* and so on.

120. *If θ be the circular measure of a positive angle less than a right angle,* $\sin\theta$ *is greater than* $\theta - \frac{\theta^3}{4}$.

For $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; and $\tan\frac{\theta}{2}$ is greater than $\frac{\theta}{2}$, therefore $\sin\frac{\theta}{2}$ is greater than $\frac{\theta}{2}\cos\frac{\theta}{2}$; therefore $\sin\theta$ is greater than $2\frac{\theta}{2}\cos^2\frac{\theta}{2}$, that is greater than $\theta\cos^2\frac{\theta}{2}$, that is greater than $\theta\left(1-\sin^2\frac{\theta}{2}\right)$. And $\sin^2\frac{\theta}{2}$ is less than $\left(\frac{\theta}{2}\right)^2$, therefore *a fortiori* $\sin\theta$ is greater than $\theta\left(1-\frac{\theta^2}{4}\right)$; that is, $\sin\theta$ is greater than $\theta - \frac{\theta^3}{4}$.

121. Thus we see that if θ lie between zero and a right angle $\sin\theta$ is less than θ and greater than $\theta - \frac{\theta^3}{4}$. And $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$.

Thus $\cos\theta$ is greater than $1 - 2\left(\frac{\theta}{2}\right)^2$, that is greater than $1 - \frac{\theta^2}{2}$.

Also $\cos\theta$ is less than $1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{32}\right)^2$, that is less than

$1 - \frac{\theta^2}{2} + \frac{\theta^4}{16} - 2\left(\frac{\theta^3}{32}\right)^2$; therefore *a fortiori* $\cos\theta$ is less than

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}.$$

122. *To calculate approximately the sine of* 10″.

The circular measure of 10″ is $\frac{10\pi}{180 \times 60 \times 60}$, that is $\frac{\pi}{64800}$; therefore the sine of 10″ is less than $\frac{\pi}{64800}$ and greater than $\frac{\pi}{64800} - \frac{1}{4}\left(\frac{\pi}{64800}\right)^3$. If we take for π the approximate value 3·141592653589793...we find $\frac{\pi}{64800} = \cdot000048481368110...$; the sine of 10″ is therefore less than this decimal fraction. And $\frac{\pi}{64800}$ is less than ·00005, therefore *a fortiori*, sin 10″ is greater than ·000048481368110...... $-\frac{1}{4}(\cdot00005)^3$; that is, sin 10″ is greater than ·000048481368078......

We have thus found two decimal fractions between which sin 10″ must lie, and these decimal fractions agree in their first twelve figures; therefore we may say that

$$\sin 10'' = \cdot000048481368......$$

and we are certain that the error is less than $\frac{1}{10^{12}}$.

The value of cos 10″ may then be found approximately since it is $\sqrt{(1 - \sin^2 10'')}$; or we may make use of the results established in Art. 121. Thus it will be found that as far as thirteen places of decimals we have

$$\cos 10'' = \cdot9999999988248......$$

123. It appears from the preceding article that as far as twelve places of decimals we have sin 10″ = the circular measure of 10″; and in the same way we may shew that sin 1″ = the circular measure of 1″ very approximately. And if n be any small number of seconds, we shall have *approximately* sin n'' = the circular measure of n'' = n times the circular measure of 1″ = $n \times \sin 1''$. Thus $n = \frac{\text{the circular measure of } n''}{\sin 1''}$ approximately; that is the number of seconds in any small angle is found approximately by

dividing the circular measure of that angle by the sine of one second.

124. We shall now shew how to calculate the sines of angles which form an arithmetical progression having $10''$ for the common difference.

Let a denote any angle, then

$$\sin(n+1)a + \sin(n-1)a = 2\sin na \cos a;$$

suppose $2\cos a = 2 - k$, then

$$\sin(n+1)a + \sin(n-1)a = (2-k)\sin na,$$

therefore $\sin(n+1)a - \sin na = \sin na - \sin(n-1)a - k\sin na.$

Now suppose $a = 10''$, then $\sin a$ is known and $\cos a$ is known, and therefore k is known; we put $n = 1$, and thus we obtain the value of $\sin 20'' - \sin 10''$, and thence the value of $\sin 20''$; next we put $n = 2$, and thus we obtain the value of $\sin 30'' - \sin 20''$, and thence the value of $\sin 30''$; next we put $n = 3$, and so on. It will be seen that the only laborious part of this operation consists in the multiplication by k of the sines as they are successively found; but from the value of $\cos 10''$ it follows that $k = \cdot 0000000023504$, and the smallness of k facilitates the process.

125. When the sines of angles up to 45° have been calculated, those for the remainder of the quadrant might be deduced by the theorem

$$\sin(45^\circ + A) - \sin(45^\circ - A) = 2\cos 45^\circ \sin A = \sqrt{2}\,.\sin A;$$

this would require the multiplication of the sines already found by the approximate value of $\sqrt{2}$. If however we calculate the sines of angles up to 60°, those for the remainder of the quadrant may be very easily found from the theorem

$$\sin(60^\circ + A) - \sin(60^\circ - A) = 2\cos 60^\circ \sin A = \sin A.$$

126. When the values of the sines of all the proposed angles in the first quadrant are known the values of the cosines are also known, for the cosine of any angle is equal to the sine of the complement of the angle. The values of the tangents can be found by

dividing the sine of every angle by the cosine of that angle. The tangents of angles greater than 45° may be easily inferred from those of angles less than 45° by the theorem

$$\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A,$$

which gives

$$\tan(45^\circ + A) = \tan(45^\circ - A) + 2\tan 2A.$$

The cotangents are known since the cotangent of any angle is equal to the tangent of the complement of the angle. The cosecants may be obtained by calculating the reciprocals of the sines; they may however be obtained more simply from the tables of tangents by the theorem

$$\operatorname{cosec} A = \frac{1}{2}\left\{\tan\frac{A}{2} + \cot\frac{A}{2}\right\}.$$

The secants are known since the secant of any angle is equal to the cosecant of the complement of the angle.

127. In the method adopted for calculating the sines of angles, the sine of 10″ was first obtained to twelve places of decimals, and then the values of sin 20″, sin 30″, &c. were deduced in succession. It will not however follow that the values of the sines of all the angles are correct to twelve places of decimals, and it is therefore useful to be able to test the extent to which the results are correct; and moreover it is essential to be able to test the correctness with which the calculations are performed. We may for this purpose compare the value of the sine of any angle obtained in the manner which has been explained with its value obtained independently. Thus, for example, we know that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$; thus the sine of 18° may easily be calculated to any degree of approximation, and by comparison with the value obtained in the tables we can judge how far we can rely upon the tables. There are however two formulæ which are usually called *formulæ of verification* from the fact that they can be easily used to verify any part of the calculated tables. These formulæ are

$$\sin A + \sin(72^\circ + A) - \sin(72^\circ - A) = \sin(36^\circ + A) - \sin(36^\circ - A),$$

$$\cos A + \cos(72^\circ + A) + \cos(72^\circ - A) = \cos(36^\circ + A) + \cos(36^\circ - A);$$

they may be readily proved; for

$$\sin(72^\circ + A) - \sin(72^\circ - A) = 2\cos 72^\circ \sin A = \frac{\sqrt{5}-1}{2}\sin A,$$

$$\sin(36^\circ + A) - \sin(36^\circ - A) = 2\cos 36^\circ \sin A = \frac{\sqrt{5}+1}{2}\sin A,$$

therefore $\sin A + \sin(72^\circ + A) - \sin(72^\circ - A) = \sin A + \frac{\sqrt{5}-1}{2}\sin A$

$$= \frac{\sqrt{5}+1}{2}\sin A = \sin(36^\circ + A) - \sin(36^\circ - A).$$

Similarly the second formula may be proved; or it may be deduced from the first by changing A into $90^\circ - A$.

Then if we ascribe any value to A, and take from the tables the values of the sines and cosines of the angles involved, these values must satisfy the *formulæ of verification* to a certain number of places of decimals, if the tables have been correctly calculated to that number of decimal places.

128. Some further remarks upon Trigonometrical Tables will be given in a subsequent chapter, in which we shall explain the method of using such tables. We will add here some theorems which will extend the results obtained in Art. 121; these theorems will furnish interesting examples although not of any immediate practical importance.

129. *The limit of* $\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\cos\frac{x}{2^n}$ *when the integer* n *is indefinitely increased is* $\frac{\sin x}{x}$.

For
$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 4\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2}$$

$$= 8\sin\frac{x}{8}\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}$$

....

$$= 2^n\sin\frac{x}{2^n}\cos\frac{x}{2^n}\ldots\ldots\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}.$$

Therefore $$\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\ldots\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\frac{x}{2^n}}.$$

Now $$\frac{\sin x}{2^n\sin\frac{x}{2^n}}=\frac{\sin x}{x}\,\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}}$$

and the limit of this when n is indefinitely increased is $\frac{\sin x}{x}$, since by Art. 118, the limit of $\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}}$ is unity. This result is sometimes cited as *Euler's Formula.*

130. *To prove that if x be the circular measure of a positive angle less than a right angle* $\sin x$ *is greater than* $x-\frac{x^3}{6}$.

By Art. 121, $\cos x$ is greater than $1-\frac{x^2}{2}$;

therefore $\cos\frac{x}{2}\cos\frac{x}{4}$ is greater than $\left(1-\frac{x^2}{2^3}\right)\left(1-\frac{x^2}{2^5}\right)$,

and *a fortiori* greater than $1-\left(\frac{x^2}{2^3}+\frac{x^2}{2^5}\right)$;

therefore $\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is greater than $\left\{1-\left(\frac{x^2}{2^3}+\frac{x^2}{2^5}\right)\right\}\left\{1-\frac{x^2}{2^7}\right\}$,

and *a fortiori* greater than $1-\left(\frac{x^2}{2^3}+\frac{x^2}{2^5}+\frac{x^2}{2^7}\right)$.

By proceeding in this way we find that

$\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\ldots\cos\frac{x}{2^n}$ is greater than

$$1-\left(\frac{x^2}{2^3}+\frac{x^2}{2^5}+\frac{x^2}{2^7}+\ldots\ldots+\frac{x^2}{2^{2n+1}}\right),$$

that is, greater than $1 - \dfrac{x^2}{2^3}\dfrac{1 - \dfrac{1}{2^{2m}}}{1 - \dfrac{1}{2^2}}$,

that is, greater than $1 - \left(\dfrac{x^2}{6} - \dfrac{x^2}{3 \,.\, 2^{2m+1}}\right)$,

and *a fortiori* greater than $1 - \dfrac{x^2}{6}$.

Hence, by Art. 129,

$$\frac{\sin x}{x} \text{ is greater than } 1 - \frac{x^2}{6},$$

therefore $$\sin x \text{ is greater than } x - \frac{x^3}{6}.$$

By proceeding as in Art. 121, we may now shew that

$$\cos x \text{ is less than } 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

(Serret's *Trigonometry*.)

MISCELLANEOUS EXAMPLES.

1. Let P be any point in a semicircle whose diameter is AB and centre C; draw PM perpendicular to AB, and draw PA, PB; from this construction, observing that the angles BPM and PAM are each equal to half of PCB, deduce the formula

$$\frac{1 - \cos A}{1 + \cos A} = \tan^2 \frac{A}{2}.$$

2. If $\cos\theta = \dfrac{a\cos\phi - b}{a - b\cos\phi}$, then $\dfrac{\tan\frac{\theta}{2}}{\sqrt{(a+b)}} = \dfrac{\tan\frac{\phi}{2}}{\sqrt{(a-b)}}$.

3. If $\tan^2\theta = 2\tan^2\phi + 1$, then $\cos 2\theta + \sin^2\phi = 0$.

4. If $\sec 2\theta = 2\sec\theta\,\mathrm{cosec}\,\theta$, then $\mathrm{cosec}\,2\theta = \mathrm{cosec}^2\theta - \sec^2\theta$.

5. If $\tan\theta = n\tan\phi$, shew that $\tan^2(\theta-\phi)$ cannot exceed $\frac{(n-1)^2}{4n}$.

6. Reduce $\sin\theta + \sin\phi - \cos\theta\sin(\theta+\phi)$ to a single term.

7. Shew that

$$\frac{\sin\beta\cos\alpha(\tan\alpha+\tan\beta)}{1-\cos(\alpha+\beta)} + \frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)} = 1.$$

8. What is approximately the height of an object which at the distance of a mile subtends at the eye an angle of one minute?

9. Find approximately the distance at which a circular plate of six inches diameter must be placed so as just to conceal the Moon, supposing the apparent diameter of the Moon to be half a degree.

10. If $\sin 3A = n\sin A$ be true for any value of A besides zero, or two right angles, or a multiple of two right angles, shew that n must lie between 3 and -1; solve the equation when $n=2$.

11. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, shew that $\tan(\alpha-\beta) = (1-n)\tan\alpha$.

12. If $\sin 3\theta$ be given, determine the number of values of $\tan\theta$.

13. Prove that $64(\cos^8 A + \sin^8 A) = \cos 8A + 28\cos 4A + 35$.

14. Find all the values of θ and ϕ which satisfy

$$\cos\theta\cos\phi + 1 = 0.$$

15. If $n^2\sin^2(\alpha+\beta) = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha-\beta)$, shew that

$$\tan\alpha = \frac{1\pm n}{1\mp n}\tan\beta.$$

16. Find the limit of $\frac{\sin 4\theta\cot\theta}{\text{vers}\, 2\theta\cot^2 2\theta}$, when θ is indefinitely diminished.

Solve the following equations:

17. $\sin\theta + \cos\theta = \sqrt{2}$.

18. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

19. $\sin 2\theta = \cos\theta$.

20. $(4-\sqrt{3})(\sec\theta + \operatorname{cosec}\theta) = 4(\sin\theta\tan\theta + \cos\theta\cot\theta)$.

21. $\cos\theta - \cos 2\theta = \sin 3\theta$.

22. $\cot\theta - \tan\theta = \cos\theta + \sin\theta$.

23. $2\sin^2\theta + \sin^2 2\theta = 2$.

24. $\tan\theta + 2\cot 2\theta = \sin\theta\left(1 + \tan\theta\tan\frac{\theta}{2}\right)$.

25. $\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{6}$.

26. $\operatorname{cosec}\theta = \operatorname{cosec}\frac{\theta}{2}$.

27. $\cos\theta\cos 3\theta = \cos 5\theta\cos 7\theta$.

28. $\sin\theta\sin 3\theta = \frac{1}{2}$.

29. $4\sin^2\theta + \sin^2 2\theta = 3$.

30. $(1-\tan\theta)(1+\sin 2\theta) = 1 + \tan\theta$.

31. $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$.

32. $\sin\theta - \cos\theta = 4\sin\theta\cos^2\theta$.

33. $(\cot\theta - \tan\theta)^2(2-\sqrt{3}) = 4(2+\sqrt{3})$.

34. $2\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)(1+\sin\theta) = 1 + \cos 2\theta$.

35. $\sin 9\theta + \sin 5\theta + 2\sin^2\theta = 1$.

X. LOGARITHMS AND LOGARITHMIC SERIES.

131. It will be necessary now for the student to become acquainted with the nature and use of logarithms, and the mode of calculating them. As it is usual to introduce into works on Trigonometry a chapter on these subjects, we shall repeat here what we have given in the *Algebra.*

132. Suppose $a^x = n$, then x is called the *logarithm of* n *to the base* a; thus the logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.

The logarithm of n to the base a is written $\log_a n$; thus $\log_a n = x$ expresses the same relation as $a^x = n$.

133. For example $3^4 = 81$; thus 4 is the logarithm of 81 to the base 3.

If we wish to find the logarithms of the numbers 1, 2, 3, ... to a given base 10, for example, we have to solve a series of equations $10^x = 1$, $10^x = 2$, $10^x = 3$,...... We shall see in some subsequent articles that this can be done *approximately*, that is, for example, although we cannot find such a value of x as will make $10^x = 2$ *exactly*, yet we can find such a value of x as will make 10^x differ from 2 by as small a quantity as we please.

We shall now prove some of the properties of logarithms.

134. *The logarithm of* 1 *is* 0 *whatever the base may be.*

For $$a^x = 1 \text{ when } x = 0.$$

135. *The logarithm of the base itself is unity.*

For $$a^x = a \text{ when } x = 1.$$

136. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For let $x = \log_a m,\ y = \log_a n;$

therefore $m = a^x,\ n = a^y;$

therefore $mn = a^x a^y = a^{x+y};$

therefore $\log_a mn = x + y = \log_a m + \log_a n.$

137. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For let $x = \log_a m,\ y = \log_a n;$

therefore $m = a^x,\ n = a^y;$

therefore $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y};$

therefore $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$

138. *The logarithm of any power, integral or fractional, of a number is equal to the product of the logarithm of the number by the index of the power.*

For let $m = a^x$; therefore $m^r = (a^x)^r = a^{xr}$,

therefore $\log_a (m^r) = xr = r \log_a m.$

139. *To find the relation between the logarithms of the same number to different bases.*

Let $x = \log_a m,\ y = \log_b m;$

therefore $m = a^x$ and $= b^y;$

therefore $a^x = b^y;$

therefore $a^{\frac{x}{y}} = b$, and $b^{\frac{y}{x}} = a;$

therefore $\frac{x}{y} = \log_a b$, and $\frac{y}{x} = \log_b a.$

Hence $y = x \log_b a$, and $= \frac{x}{\log_a b}.$

Hence the logarithm of a number to the base b may be found by multiplying the logarithm of the number to the base a by

$$\log_b a, \text{ or by } \frac{1}{\log_a b}.$$

We may notice that $\log_b a \times \log_a b = 1.$

140. In practical calculations the only base that is used is 10; logarithms to the base 10 are called *common* logarithms. We will point out in the next two articles some peculiarities which constitute the advantage of the base 10. We shall require the following definition; the integral part of any logarithm is called the *characteristic*, and the decimal part the *mantissa*.

141. In the common system of logarithms, if the logarithm of any number be known we can immediately determine the logarithm of the product or quotient of that number by any power of 10.

For $$\log_{10} 10^n \times N = \log_{10} N + \log_{10} 10^n = \log_{10} N + n,$$

$$\log_{10} \frac{N}{10^n} = \log_{10} N - \log_{10} 10^n = \log_{10} N - n.$$

That is, if we know the logarithm of any number we can determine the logarithm of any number which has the same figures, but differs merely by the position of the decimal point.

142. In the common system of logarithms the characteristic of the logarithm of any number can be determined by inspection.

For suppose the number to be greater than unity and to lie between 10^n and 10^{n+1}; then its logarithm must be greater than n and less than $n+1$; hence the characteristic of the logarithm is n.

Next suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$; then its logarithm will be some negative quantity between $-n$ and $-(n+1)$; hence if we agree that the *mantissa shall always be positive*, the characteristic will be $-(n+1)$.

We shall now proceed to investigate formulæ for the calculation of logarithms.

143. *To expand* a^x *in a series of ascending powers of* x; *that is, to expand a number in a series of ascending powers of its logarithm to a given base.*

$$a^x = \{1 + (a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{1 \cdot 2}(a-1)^2$$
$$+ \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}(a-1)^3 + \frac{x(x-1)(x-2)(x-3)}{1 \cdot 2 \cdot 3 \cdot 4}(a-1)^4 + \ldots\ldots$$
$$= 1 + x\{a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots\ldots\}$$
$$+ \text{ terms involving } x^2, x^3, \text{ \&c.}$$

This shews that a^x *can* be expanded in a series beginning with 1 and proceeding in ascending powers of x; we may therefore suppose that

$$a^x = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots\ldots$$

where c_1, c_2, c_3 ...... are quantities which do not depend on x, and which therefore remain unchanged however x may be changed; also

$$c_1 = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots\ldots$$

while c_2, c_3, ... are at present unknown; we proceed to find their values. Changing x into $x + y$ we have

$$a^{x+y} = 1 + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + \ldots\ldots;$$

but $$a^{x+y} = a^x a^y = a^y\{1 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots\}.$$

Since the two expressions for a^{x+y} are identically equal, we may *assume* that the coefficients of x in the two expressions are equal, thus

$$c_1 + 2c_2 y + 3c_3 y^2 + 4c_4 y^3 + \ldots\ldots = c_1 a^y$$
$$= c_1\{1 + c_1 y + c_2 y^2 + c_3 y^3 + \ldots\ldots\}.$$

In this identity we may *assume* that the coefficients of the corresponding powers of y are equal; thus

$$2c_2 = c_1^2; \qquad \text{therefore, } c_2 = \frac{c_1^2}{2},$$

$$3c_3 = c_1 c_2; \qquad \text{therefore, } c_3 = \frac{c_1 c_2}{3} = \frac{c_1^3}{1 \cdot 2 \cdot 3},$$

$$4c_4 = c_1 c_3; \qquad \text{therefore, } c_4 = \frac{c_1 c_3}{4} = \frac{c_1^4}{1 \cdot 2 \cdot 3 \cdot 4}.$$

....

Thus, $$a^x = 1 + c_1 x + \frac{c_1^2 x^2}{\underline{|2}} + \frac{c_1^3 x^3}{\underline{|3}} + \frac{c_1^4 x^4}{\underline{|4}} + \ldots\ldots$$

Since this result is true for all values of x, take x such that $c_1x=1$, then $x=\dfrac{1}{c_1}$ and

$$a^{\frac{1}{c_1}}=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\ldots;$$

this series is usually denoted by e; thus $a^{\frac{1}{c_1}}=e$, therefore $a=e^{c_1}$ and $c_1=\log_e a$; hence,

$$a^x=1+(\log_e a)\,x+\frac{(\log_e a)^2\,x^2}{\lfloor 2}+\frac{(\log_e a)^3\,x^3}{\lfloor 3}+\ldots\ldots$$

This result is called the *Exponential Theorem.*

Put e for a, then $\log_e a$ becomes $\log_e e$, that is, unity, (Art. 135); thus,

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ldots\ldots$$

With respect to the *assumption* which has been made twice in the course of this article, the student is referred to the chapter on *Indeterminate Coefficients* in the *Algebra.*

144. By actual calculation we may find approximately the numerical value of the series which we have denoted by e; it is 2·718281828......

145. *To expand* $\log_e(1+x)$ *in a series of ascending powers of* x.

We have seen in Art. 143, that $c_1=\log_e a$; that is, by the same Article,

$$\log_e a=a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots\ldots$$

For a put $1+x$; hence,

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

This series may be applied to calculate $\log_e(1+x)$ if x is a proper fraction; but unless x be very small, the terms diminish so slowly that we shall have to retain a large number of them; if x be greater than unity the series is altogether unsuitable. We shall therefore deduce some more convenient formulæ.

146. We have

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots$$

therefore $$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots,$$

by subtraction we obtain the value of $\log_e(1+x) - \log_e(1-x)$, that is, of $\log_e \frac{1+x}{1-x}$;

therefore $$\log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\right\}\dots\dots$$

In this series write $\frac{m-n}{m+n}$ for x, and therefore $\frac{m}{n}$ for $\frac{1+x}{1-x}$ thus

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\}\dots\dots(1).$$

Put $n = 1$, then

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \dots\right\}\dots\dots(2).$$

Again in (1) put $m = n + 1$, thus we obtain the value of $\log_e \frac{n+1}{n}$; therefore

$$\log_e(n+1) - \log_e n$$
$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\dots\right\}\dots\dots\dots(3).$$

147. The series (2) of the preceding article will enable us to find $\log_e 2$; put $m = 2$, then by calculation we shall find

$$\log_e 2 = \cdot6931471\dots\dots$$

From the series (3) we can calculate the logarithm of either of two consecutive numbers when we know that of the other. Put

$n=2$, and by making use of the known value of $\log_e 2$, we shall obtain

$$\log_e 3 = 1\cdot0986122\ldots\ldots$$

Put $n=9$ in (3); then $\log_e 9 = \log_e 3^2 = 2\log_e 3$ and is therefore known; hence we shall find

$$\log_e 10 = 2\cdot3025850\ldots\ldots$$

Logarithms to the base e are called *Napierian* logarithms, from Napier the inventor of logarithms; they are also called *natural* logarithms, being those which occur first in our investigation of a method of calculating logarithms. We have said that the base 10 is the only base used in the practical application of logarithms, but logarithms to the Napierian base occur frequently in theoretical investigations.

148. From Art. 139 we see that the logarithm of a number to the base 10 can be found by multiplying the Napierian logarithm by $\frac{1}{\log_e 10}$, that is, by $\frac{1}{2\cdot30258509}$, or by $\cdot43429448$; this multiplier is called the *modulus* of the common system.

The series in Art. 146 may be so adjusted as to give common logarithms; for example, take the series (3), multiply throughout by the modulus which we shall denote by μ; thus

$$\mu\log_e(n+1)-\mu\log_e n = 2\mu\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\};$$

that is,

$$\log_{10}(n+1)-\log_{10} n = 2\mu\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\}.$$

Similarly from Art. 145 we have

$$\log_{10}(1+x)=\mu\left\{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right\}.$$

149. *The quantity* e *is incommensurable.*

For suppose if possible $e=\frac{m}{n}$, where m and n are integers; thus

$$\frac{m}{n}=2+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots$$

multiply both sides by $\underline{|n}$; then

$$m\,\underline{|n-1}=\text{an integer}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

But $$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

is a fraction, for it is greater than $\frac{1}{n+1}$ and less than the geometrical progression

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots$$

that is, less than $\frac{1}{n}$.

Thus the difference of two integers is equal to a fraction, which is absurd. Therefore e is incommensurable.

150. We will conclude this chapter by investigating two limits which will be useful hereafter.

To find the limit of $\left(\cos\frac{a}{n}\right)^n$ *when* n *is increased indefinitely.*

Let $$u=\left(\cos\frac{a}{n}\right)^n=\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}};\ \text{then}$$

$$\log u=\log\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}}=\frac{n}{2}\log\left(1-\sin^2\frac{a}{n}\right)$$

$$=-\frac{n}{2}\left(\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}+\ldots\ldots\right).$$

Now $n \sin \frac{a}{n} = a \dfrac{\sin \frac{a}{n}}{\frac{a}{n}} = a$ when n is increased indefinitely (Art. 118); therefore $n \sin^2 \frac{a}{n} = a \sin \frac{a}{n} = 0$ ultimately; and similarly $n \sin^4 \frac{a}{n}$, $n \sin^6 \frac{a}{n}$, ... vanish ultimately. Therefore $\log u = 0$; therefore $u = 1$. Thus the required limit is unity.

To find the limit of $\left(\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}\right)^n$ *when* n *is increased indefinitely.*

We know by Art. 116 that $\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}$ is less than 1 and greater than $\dfrac{\sin \frac{a}{n}}{\tan \frac{a}{n}}$, that is, greater than $\cos \frac{a}{n}$; hence $\left(\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}\right)^n$ is less than 1^n or 1 and greater than $\left(\cos \frac{a}{n}\right)^n$; and by the preceding article the limit of $\left(\cos \frac{a}{n}\right)^n$ is unity, therefore the limit of $\left(\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}\right)^n$ is unity.

MISCELLANEOUS EXAMPLES.

1. Find the logarithm of 128 to the base $\sqrt[3]{4}$.

2. Find the logarithm of $243 \sqrt[3]{9}$ to the base $\sqrt{3}$.

3. Find the following logarithms, $\log_3 2187$, $\log_{10} \cdot 0001$, $\log_2 \cos 45^\circ$.

4. Find approximately the value of x from the equation

$$5^{6-4x} = 2^{x+8},$$

having given $\log 2 = \cdot 301030$.

5. Given $\log \cdot 224 = a$ and $\log 125 = b$, find $\log 2$ and $\log 7$.

6. Required the characteristics of $\log_6 725$, and of $\log_6 \sqrt[5]{(\cdot 0725)}$.

7. Given $\log 2 = \cdot 301030$, $\log 405 = 2 \cdot 607455$, find $\log \cdot 003$.

8. Given $\log 2 = \cdot 30103$, $\log 7 = \cdot 845098$, find $\log 98$ and $\log \left(\frac{4}{343}\right)^{\frac{1}{2}}$.

9. Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 47712$, find $\log (\cdot 0020736)^{\frac{1}{3}}$.

10. Determine the sum of the series

$$\frac{2}{\lfloor 3} + \frac{4}{\lfloor 5} + \frac{6}{\lfloor 7} + \dots \textit{ad inf.}$$

11. Shew that

$$\frac{e}{2} = \frac{1}{\lfloor 2} + \frac{1+2}{\lfloor 3} + \frac{1+2+3}{\lfloor 4} + \frac{1+2+3+4}{\lfloor 5} + \dots \textit{ad inf.}$$

Find x from the following six equations:

12. $4 \sin x \sin (x - a) = 2 \cos a - 1$.

13. $\cos \beta \sqrt{(a^2 - x^2)} + a \sin a = x \sin \beta$.

14. $\sin a + \sin (x - a) + \sin (2x + a) = \sin (x + a) + \sin (2x - a)$.

15. $\cos \left(x + \frac{3}{2}\right) a + \cos \left(x + \frac{1}{2}\right) a = \sin a$.

16. $x^2 \cos a \cos \left(a - \frac{\beta}{2}\right) + x \cos (a - \beta) = 2 \cos \frac{\beta}{2}$.

17. $\cot 2^{n-1}\alpha - \cot 2^n\alpha = \operatorname{cosec} 3\alpha$.

18. Solve the equation $m \operatorname{vers} \theta = n \operatorname{vers} (\alpha - \theta)$.

19. Solve the equation $\cos n\theta + \cos (n-2)\theta = \cos \theta$.

20. Solve the following equation, and shew that there are seven positive values of θ greater than 0 and less than 2π,

$$\sin \theta + \sin 3\theta = \sin 2\theta + \sin 4\theta.$$

21. Find $\tan x$ from the equation $\tan x = \tan \beta \tan (\alpha + x)$; and shew that in order that $\tan x$ may be real, $\tan \beta$ must not lie between $(\sec \alpha - \tan \alpha)^2$ and $(\sec \alpha + \tan \alpha)^2$.

22. Find the least value of θ which satisfies

$$\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{\pi}{4} + \theta\right) = \left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}}.$$

23. Given $\sin^2(n+1)\theta = \sin^2 n\theta + \sin^2(n-1)\theta$ where $(n+1)\theta$, $n\theta$, and $(n-1)\theta$ are the angles of a triangle, find an integral value of n.

24. Reduce to its simplest form and solve the equation

$$\cos^2\theta - \cos^2\alpha = 2\cos^3\theta(\cos\theta - \cos\alpha) - 2\sin^3\theta(\sin\theta - \sin\alpha).$$

25. Shew that all the angles which have the same sine as α are included in the formula $\left(2n + \frac{1}{2}\right)\pi \pm \left(\frac{\pi}{2} - \alpha\right)$.

26. Shew that all the angles which have the same cosine as α are included in the formula $\left(n + \frac{1}{2}\right)\pi + (-1)^n\left(\alpha - \frac{\pi}{2}\right)$.

27. The ambiguity $\pm$ in the formula

$$\cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{(1 - \sin A)}$$

may be replaced by $(-1)^m$, where m is the greatest integer contained in $\frac{270+A}{360}$, A being expressed in degrees.

28. The ambiguity $\pm$ in the formula

$$\tan\frac{A}{2} = \frac{\pm\sqrt{(1+\tan^2 A)}-1}{\tan A}$$

may be replaced by $(-1)^m$, where m is the greatest integer contained in $\frac{90+A}{180}$, A being expressed in degrees.

29. If $\tan(\cot x) = \cot(\tan x)$, shew that the real values of x are given by $\sin 2x = \frac{4}{(2n+1)\pi}$, where n is any integer except -1.

30. Shew how to express $\cos\frac{A}{2^n}$ in terms of $\cos A$, where n is any positive integer.

31. From the equation $\cos x = \sqrt{\frac{1+\cos 2x}{2}}$ deduce the formula for $\sin x$ in terms of $\sin 2x$, and shew how the proper signs for the radicals may be determined.

32. If the expression

$$\frac{A\cos(\theta+\alpha)+B\sin(\theta+\beta)}{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)}$$

retain the same value for all values of θ, then will

$$AA' - BB' = (A'B - AB')\sin(\alpha-\beta).$$

33. If the sum of two angles is given, shew that the sum of their sines is greatest and the sum of their tangents is least when the angles are equal.

34. If $A+B+C=90^\circ$, shew that unity is the least value of $\tan^2 A + \tan^2 B + \tan^2 C$.

35. If $A+B+C=180°$, shew that unity is the least value of $\cot^2 A+\cot^2 B+\cot^2 C$.

36. If $A+B+C=180°$, then

$2\cot A+2\cot B+2\cot C$ is greater than

$$\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C.$$

37. Shew that the sum of the three acute angles which satisfy the equation $\cos^2 A+\cos^2 B+\cos^2 C=1$ is less than $180°$.

38. If each of the angles A, B, C be less than $90°$, then $\sin(A+B+C)$ is less than $\sin A+\sin B+\sin C$.

39. Find the limit of $\left(\cos\frac{a}{n}\right)^{n^2}$ when n is increased indefinitely.

40. Find the limit of $\left(\cos\frac{a}{n}\right)^{n^3}$ when n is increased indefinitely.

41. Shew that $\sin\theta$ is greater than $\tan\theta-\frac{\tan^3\theta}{2}$.

XI. USE OF LOGARITHMIC AND TRIGONOMETRICAL TABLES.

151. In the preceding two chapters we have shewn how tables of the values of the Trigonometrical Ratios may be calculated and how tables of logarithms may be calculated, and we shall now shew how to use such tables; we begin with tables of logarithms. It is obvious that tables of logarithms may be calculated to various degrees of approximation; they may be calculated to 5, 6, 7 or a higher number of decimal places. For a list of logarithmic and trigonometrical tables the student may consult

the articles *Tables* in the *Penny Cyclopædia* and its *Supplement.* Different tables present some variety in their mode of arrangement, and are usually accompanied with full explanation of their peculiarities and the methods of using the tables; we shall not enter into any minute account of the way in which tables may be used with the greatest advantage, but shall give such general illustrations as will enable the student to avail himself of any set of tables for the purpose of occasional calculation. The logarithms will always be supposed taken to the base *ten.*

152. We may observe that throughout all *approximate* calculations it is usual to take for the last figure which we retain, the figure which gives the nearest approach to the true value. Thus for example, suppose we have the decimal fraction ·3726; if we wish to retain only three places of decimals we should write ·373 and not ·372; the former is too large and the latter too small, but the excess in the former case is ·0004, and the defect in the latter case is ·0006, so that there is a smaller error in the former case than in the latter case. Thus we have this general rule, when only a certain number of decimal places is to be retained—*strike off the rest of the figures and increase the last figure retained by* 1 *if the first figure struck off be 5 or greater than 5.*

We now proceed to explain the use of tables of common logarithms; and we shall use tables of *seven* places of decimals.

153. *To find the logarithm of a given number.*

If the number be contained in the Table we have merely to take the decimal part of the logarithm immediately from the Table and prefix the *characteristic* (Art. 142). For example, required the logarithm of 534. The table gives ·7275413 as the decimal part, and the characteristic is 2; therefore

$$\log 534 = 2{\cdot}7275413.$$

Similarly,

$$\log 53400 = 4{\cdot}7275413,$$

$$\log {\cdot}0534 = \bar{2}{\cdot}7275413.$$

In the last example the characteristic is -2, and this is denoted by the bar placed over the 2.

Suppose, however, that the given number is not contained in the Table; the Table for instance may give the logarithms of numbers from 1 up to 100000 and we may require the logarithm of 5340234. Here we can take from the Table the logarithm of 5340200, and the logarithm of 5340300; we have

$$\log 5340300 = 6{\cdot}7275657$$
$$\log 5340200 = 6{\cdot}7275575$$
$$\text{difference} = {\cdot}0000082$$

The required logarithm of course lies between the two logarithms which we have taken from the Table. Now we see that corresponding to the increase 100 in the number there is an increase ·0000082 in the logarithm; and we *assume* that corresponding to an increase 34 in the number there will be a *proportional* increase in the logarithm. Let x denote the quantity which we must add to the logarithm of 5340200 in order to obtain the logarithm of 5340234; then we have from the assumption which we have made the following proportion:

$$100 : 34 :: {\cdot}0000082 : x;$$

$$\text{therefore } x = \frac{34}{100} \times {\cdot}0000082 = {\cdot}0000028 \text{ (Art. 152)};$$

$$\text{therefore } \log 5340234 = 6{\cdot}7275575 + {\cdot}0000028 = 6{\cdot}7275603.$$

154. We assumed in the preceding article that the increase in a logarithm is *proportional* to the increase in the number; this is a case of what is called the *principle of proportional parts*, and although it is not *strictly* true, yet it is in most cases sufficient for practical purposes. We shall in the next chapter investigate the subject, and shew to what degree of approximation we can rely upon the *principle of proportional parts*.

155. The process given in Art. 153 is facilitated in large Tables in the following manner. Required the logarithm of 23453487.

$$\log 23454000 = 7\cdot3702169$$
$$\log 23453000 = 7\cdot3701984$$
$$\text{difference} = \cdot0000185$$

Proportional Parts.

1	185
2	370
3	555
4	740
5	925
6	1110
7	1295
8	1480
9	1665

Here by the process of Art. 153 we have to multiply $\cdot0000185$ by $\frac{487}{1000}$, that is, by $\frac{4}{10} + \frac{8}{100} + \frac{7}{1000}$. Now the multiplication is effected for us, and the results given in a small Table headed *Proportional parts*, which is printed on the same page as the two logarithms which we have taken from the Table; the small Table shews that $4 \times \cdot0000185 = \cdot0000740$, $8 \times \cdot0000185 = \cdot0001480$, $7 \times \cdot0000185 = \cdot0001295$; and from these results, by dividing by 10, 100 and 1000 respectively, we obtain the three parts which we require. The process may be arranged thus:

log 23453000 =		7·3701984
add for	4	740
	8	1480
	7	1295
		7·3702074095

therefore, retaining 7 places of decimals,

$$\log 23453487 = 7\cdot3702074.$$

156. We have taken as our example a *whole* number; if a decimal fraction, or a mixed quantity formed of a whole number and decimal fraction, be given, we may throw aside the decimal point, and find the decimal part of the logarithm of the whole number thus obtained; then by prefixing the proper characteristic we have the required logarithm. Thus, for example, required the logarithm of $\cdot23453487$ and of $234\cdot53487$. The decimal part of the logarithm is $\cdot3702074$; therefore

$$\log \cdot23453487 = \bar{1}\cdot3702074$$
$$\log 234\cdot53487 = 2\cdot3702074.$$

157. *To find the number which corresponds to a given logarithm.*

If the decimal part of the logarithm be found in the Table, we have merely to take the number which corresponds to it, and put the decimal point in the number in the place indicated by the characteristic. For example, required the number which has for its logarithm $\bar{2}\cdot7275413$. Corresponding to the decimal part $\cdot7275413$ we find in the Table the number 534, and as the characteristic is $\bar{2}$, there must be one cypher before the first significant figure (Art. 142); therefore the number which has the given logarithm is $\cdot0534$.

Suppose, however, that the decimal part of the given logarithm is not contained exactly in the Table; for example, let the given logarithm be $\bar{1}\cdot3702074$, we shall find that the decimal part of this logarithm is not in the Table; we have, however, corresponding to the number 23454 the decimal part of the logarithm $\cdot3702169$, and corresponding to the number 23453 the decimal part of the logarithm $\cdot3701984$; thus

$$\begin{aligned} \log 23454 &= 4\cdot3702169 \\ \log 23453 &= 4\cdot3701984 \\ \hline \text{difference} &= \cdot0000185 \end{aligned}$$

The excess of the given decimal part of the logarithm above $\cdot3701984$ is $\cdot3702074 - \cdot3701984$, that is $\cdot0000090$. The required number of course lies between $\cdot23454$ and $\cdot23453$; let d denote its excess above $\cdot23453$, then *assuming* that the increase of the number is proportional to the increase of the logarithm, we have

$$\cdot0000185 : \cdot0000090 :: 1 : d;$$

$$\text{therefore } d = \frac{90}{185} = \cdot486.$$

Therefore $\log 23453\cdot486 = 4\cdot3702074$,

and $\log \cdot23453486 = \bar{1}\cdot3702074$;

thus the required number is $\cdot23453486$.

158. We may save the labour of dividing 90 by 185 in the preceding example by means of the Table of Proportional parts given in Art. 155; the process of division, if performed, will stand thus:

$$\begin{array}{r}
185)90{\cdot}0(486 \\
740 \\
\hline
1600 \\
1480 \\
\hline
1200 \\
1110 \\
\hline
\end{array}$$

Now the products 740, 1480, 1110, are furnished ready in the Table referred to, so that we need only perform the subtractions and put down the following steps:

$$\begin{array}{cr}
 & 90 \\
4 & 740 \\
\hline
 & 160 \\
8 & 1480 \\
\hline
 & 1200 \\
6 & 1110 \\
\hline
\end{array}$$

159. We will now give some examples of the use of logarithms. Required the product of 3670·257 and 12·61158.

$$\begin{array}{rr}
\text{Log } 3670{\cdot}2 & = 3{\cdot}5646897 \\
5 & 60 \\
7 & 8 \\
\hline
\text{Log } 3670{\cdot}257 & = 3{\cdot}5646965 \\
\text{Log } 12{\cdot}611 & = 1{\cdot}1007495 \\
5 & 172 \\
8 & 28 \\
\hline
\text{Log } 12{\cdot}61158 & = 1{\cdot}1007695 \\
 & 3{\cdot}5646965 \\
\hline
\text{by adding the logs} & 4{\cdot}6654660
\end{array}$$

$$\begin{array}{lr}
\text{Decimal part of log } 46287 & \cdot 6654590 \\
\hline
 & 70 \\
7 & 66 \\
\hline
4 & 40 \\
\hline
4628774 &
\end{array}$$

Thus the required number is 46287·74, the position of the decimal point being determined by the characteristic 4.

160. Required the quotient of ·1234567 by 54·87645.

$$\begin{array}{lr}
\text{Log } \cdot 12345 = & \bar{1}\cdot 0914911 \\
6 & 211 \\
7 & 25 \\
\hline
\text{Log } \cdot 1234567 = & \bar{1}\cdot 0915147 \\
\text{Log } 54\cdot 876 = & 1\cdot 7393824 \\
4 & 32 \\
5 & 4 \\
\hline
\text{Log } 54\cdot 87645 = & 1\cdot 7393860 \\
 & \bar{1}\cdot 0915147 \\
 & 1\cdot 7393860 \\
\hline
\text{by subtracting} & \bar{3}\cdot 3521287
\end{array}$$

$$\begin{array}{lr}
\text{Decimal part of log } 22497 = & \cdot 3521246 \\
\hline
2 & 41 \\
2 & 38 \\
\hline
2249722 & 30
\end{array}$$

Thus the required number is ·002249722; there are two cyphers before the first significant figure, because the characteristic of the logarithm is $\bar{3}$.

161. Required the cube of ·3180236.

$$
\begin{array}{lrr}
\text{Log } \cdot 31802 & = & \bar{1}\cdot 5024544 \\
 3 & & 41 \\
 6 & & 8 \\
\hline
\text{Log } \cdot 3180236 & = & \bar{1}\cdot 5024593 \\
 & & 3 \\
\hline
 & & \bar{2}\cdot 5073779 \\
\text{Decimal part of log } 32164 & = & \cdot 5073701 \\
\hline
 & & 78 \\
 5 & & 67 \\
\hline
 8 & & 110 \\
\hline
3216458 & &
\end{array}
$$

Thus the required number is ·03216458.

162. Required the cube root of ·3663265.

$$
\begin{array}{lrr}
\text{Log } \cdot 36632 & = & \bar{1}\cdot 5638606 \\
 6 & & 71 \\
 5 & & 6 \\
\hline
\text{Log } \cdot 3663265 & = & \bar{1}\cdot 5638683
\end{array}
$$

We have now to divide $\bar{1}\cdot 5638683$ by 3; that is, we have to divide $-1+\cdot 5638683$ by 3. It is convenient to write the number to be divided thus, $-3+2\cdot 5638683$; then by dividing by 3 we obtain $-1+\cdot 8546228$, that is, $\bar{1}\cdot 8546228$.

$$
\begin{array}{lrr}
 & & \bar{1}\cdot 8546228 \\
\text{Decimal part of log } 71552 & = & \cdot 8546218 \\
\hline
 0 & & 10 \\
 2 & & 100 \\
\hline
7155202 & &
\end{array}
$$

Thus the required number is ·7155202.

We now proceed to the use of Trigonometrical Tables.

163. *To find the sine of a given angle.*

If the given angle be one which is contained in the Table of the sines of angles the required sine is furnished immediately by the Table; we proceed then to the case when the given angle lies between two which are contained in the Table. For example, required the sine of 44° 35′ 25″, having given from the Table

$$\sin 44^\circ\, 36' = \cdot 7021531$$
$$\sin 44^\circ\, 35' = \cdot 7019459$$
$$\text{difference} = \cdot 0002072$$

The required sine of course lies between the two sines which we have taken from the Table; let x denote its excess above the sine of 44° 35′, and assume that the increase of the sine is proportional to the increase of the angle, therefore

$$60'' : 25'' :: \cdot 0002072 : x,$$

$$\text{therefore} \qquad x = \frac{25}{60} \times \cdot 0002072 = \cdot 0000863.$$

Therefore $\sin 44^\circ\, 35'\, 25'' = \cdot 7019459 + \cdot 0000863 = \cdot 7020322.$

We have thus again assumed the *principle of proportional parts*, and we shall assume it throughout the present chapter, reserving the investigation of it for the following chapter.

164. *To find the angle which corresponds to a given sine.*

If the given sine be found in the Table the required angle is furnished immediately by the Table; we proceed then to the case when the given sine lies between two which are contained in the Table. For example, required the angle which has for its sine ·6970886, having given from the Table

$$\sin 44^\circ\, 12' = \cdot 6971651$$
$$\sin 44^\circ\, 11' = \cdot 6969565$$
$$\text{difference} = \cdot 0002086$$

The excess of the given sine above the sine of 44° 11′ is

$$\cdot 6970886 - \cdot 6969565, \text{ that is, } \cdot 0001321.$$

The required angle of course lies between the two angles which we have taken from the Table; let n be the number of seconds in its excess above 44° 11′, then

$$\cdot 0002086 : \cdot 0001321 :: 60 : n,$$

therefore $$n = 60 \times \frac{\cdot 0001321}{\cdot 0002086} = \frac{60 \times 1321}{2086} = 38.$$

Therefore the required angle is 44° 11′ 38″.

165. *To find the cosine of a given angle.*

If the given angle be one which is contained in the Table of the cosines of angles, the required cosine is furnished immediately by the Table; we proceed then to the case when the given angle lies between two which are contained in the Table. For example, required the cosine of 44° 35′ 25″, having given from the Table

$$\cos 44^\circ\, 35' = \cdot 7122303$$
$$\cos 44^\circ\, 36' = \cdot 7120260$$
$$\text{difference} = \cdot 0002043$$

Since in the first quadrant the cosine *decreases* as the angle increases, the required cosine will be less than the cosine of 44° 35′, and the required cosine of course lies between the two cosines which we have taken from the Table; let x denote its defect below the cosine of 44° 35′, then

$$60 : 25 :: \cdot 0002043 : x,$$

therefore $$x = \frac{25}{60} \times \cdot 0002043 = \cdot 0000851.$$

Therefore $$\cos 44^\circ\, 35'\, 25'' = \cdot 7122303 - \cdot 0000851 = \cdot 7121452.$$

166. *To find the angle which corresponds to a given cosine.*

If the given cosine be found in the Table the required angle is furnished immediately by the Table; we proceed then to the case when the given cosine lies between two which are contained in the Table. For example, required the angle which has for its cosine ·7169848, having given from the Table

$$\cos 44^\circ\, 11' = \cdot 7171134$$
$$\cos 44^\circ\, 12' = \cdot 7169106$$
$$\text{difference} = \cdot 0002028$$

The given cosine falls short of the cosine of 44° 11′ by ·7171134 − ·7169848, that is, by ·0001286; the required angle of course lies between the two angles which we have taken from the Table; let n be the number of seconds in its excess above 44° 11′, then

$$\cdot0002028 : \cdot0001286 :: 60 : n,$$

$$\text{therefore } n = 60 \times \frac{\cdot0001286}{\cdot0002028} = \frac{60 \times 1286}{2028} = 38.$$

Therefore the required angle is 44° 11′ 38″.

167. It will not be necessary to give examples for the other Trigonometrical Functions; the important fact to be remembered is that in the first quadrant the tangent and secant *increase* as the angle increases, and the cotangent and cosecant *decrease* as the angle increases; thus the tangent and secant are treated in the same way as the *sine*, and the cotangent and cosecant in the same way as the *cosine*.

168. The Tables of Trigonometrical Functions which we have hitherto considered are called Tables of the *natural* Functions to distinguish them from other Tables which we now proceed to consider. The Table of sines of angles for example is called a Table of *natural* sines; if we take the *logarithms of the sines* of all the angles which have been calculated we form a new Table which is called a Table of *Logarithmic* sines. Similarly, we can form a Table of the logarithms of the cosines of angles, and a Table of the logarithms of the tangents of angles, and so on; these Tables are called respectively Tables of *logarithmic cosines*, Tables of *logarithmic tangents*, and so on.

169. The great advantage which we obtain from these Logarithmic Tables is that calculations are much abbreviated with their assistance; this is especially the case, as we shall see hereafter, in what is called the *solution of Triangles*. We have stated as sufficiently obvious that these Logarithmic Tables may be calculated by taking the logarithms of the values of the Trigonometrical Functions which have been already tabulated; it will be shewn however in

the higher parts of the subject that the *Logarithmic* Tables can be calculated *independently*, that is, without the use of the Tables of the *natural* Functions. We proceed now to exemplify the use of the Tables of Logarithmic Functions.

170. Since the sine of an angle is never greater than unity the logarithm of the sine will never be a *positive* quantity; also the same remark is true for the cosine. The logarithm of the tangent of an angle will be negative if the angle be less than 45°, and the logarithm of the cotangent of an angle will be negative if the angle be greater than 45°. In order to avoid the occurrence of negative quantities in the Tables it is found convenient to add 10 to the logarithm of every Trigonometrical Function before registering it in the Tables; the logarithm so increased is called the *Tabular logarithm* and is usually denoted by the letter L. Thus $L \sin A$ means the *Tabular logarithm* of the sine of A, and it is equal to the real logarithm of the sine of A increased by ten. Of course in calculations we shall have to remember and to allow for this increase of the real logarithms; this will be seen when we come to the *solution of Triangles*. In what follows we shall exemplify the use of the Tables of Logarithmic Functions.

171. *To find the tabular logarithmic sine of a given angle.*

If the given angle be one which is contained in the Table of the logarithmic sines the required result is furnished immediately by the Table; we proceed then to the case when the given angle lies between two which are contained in the Table. For example, required the tabular logarithmic sine of $44^\circ\,35'\,25''\cdot7$, having given from the Table

$$L \sin 44^\circ\,35'\,30'' = 9\cdot8463678$$
$$L \sin 44^\circ\,35'\,20'' = 9\cdot8463464$$
$$\text{difference} = \cdot0000214$$

The required tabular logarithmic sine lies of course between the two which we have taken from the Table; let x denote its excess above the tabular logarithmic sine of $44^\circ 35' 20''$; then by the *principle of proportional parts*

$$10 : 5\cdot7 :: \cdot0000214 : x,$$

$$\text{thus } x = \frac{5\cdot7}{10} \times \cdot0000214 = \cdot0000122.$$

Therefore $L \sin 44^\circ 35' 25''\cdot7 = 9\cdot8463464 + \cdot0000122 = 9\cdot8463586$.

172. *To find the angle which corresponds to a given tabular logarithmic sine.*

If the given tabular logarithmic sine be found in the Table the required angle is furnished immediately by the Table; we proceed then to the case when the given tabular logarithmic sine lies between two which are contained in the Table. For example, required the angle which has for its tabular logarithmic sine $9\cdot8432894$, having given from the Table

$$L \sin 44^\circ 11' 40'' = 9\cdot8432923$$

$$L \sin 44^\circ 11' 30'' = 9\cdot8432707$$

$$\text{difference} = \cdot0000216$$

The excess of the given tabular logarithmic sine above that of $44^\circ 11' 30''$ is $9\cdot8432894 - 9\cdot8432707$, that is, $\cdot0000187$. The required angle of course lies between the two angles which we have taken from the Table; let n be the number of seconds in its excess above $44^\circ 11' 30''$, then

$$\cdot0000216 : \cdot0000187 :: 10 : n,$$

therefore $$n = 10 \times \frac{\cdot0000187}{\cdot0000216} = \frac{10 \times 187}{216} = 8\cdot7.$$

Therefore the required angle is $44^\circ 11' 38''\cdot7$.

173. *To find the tabular logarithmic cosine of a given angle.*

If the given angle be one which is contained in the Table of the logarithmic cosines the required result is furnished immediately by the Table; we proceed then to the case when the given angle lies between two which are contained in the Table. For example, required the tabular logarithmic cosine of $44^\circ\ 35'\ 25''\cdot 7$, having given from the Table

$$L\cos 44^\circ\ 35'\ 20'' = 9\cdot 8525789$$
$$L\cos 44^\circ\ 35'\ 30'' = 9\cdot 8525582$$
$$\text{difference} = \cdot 0000207$$

The required tabular logarithmic cosine lies of course between the two which we have taken from the Table, and is *less* than the tabular logarithmic cosine of $44^\circ\ 35'\ 20''$; let x denote its defect below the latter; then

$$10 : 5\cdot 7 :: \cdot 0000207 : x,$$

$$\text{thus } x = \frac{5\cdot 7}{10} \times \cdot 0000207 = \cdot 0000118.$$

Therefore $L\cos 44^\circ\ 35'\ 25''\cdot 7 = 9\cdot 8525789 - \cdot 0000118 = 9\cdot 8525671.$

174. *To find the angle which corresponds to a given tabular logarithmic cosine.*

If the given tabular logarithmic cosine be found in the Table the required angle is furnished immediately by the Table; we proceed then to the case when the given tabular logarithmic cosine lies between two which are contained in the Table. For example, required the angle which has for its tabular logarithmic cosine $9\cdot 8555086$, having given from the Table

$$L\cos 44^\circ\ 11'\ 30'' = 9\cdot 8555264$$
$$L\cos 44^\circ\ 11'\ 40'' = 9\cdot 8555060$$
$$\text{difference} = \cdot 0000204$$

The given tabular logarithmic cosine falls short of that of $44^\circ\ 11'\ 30''$ by $9\cdot 8555264 - 9\cdot 8555086$, that is, $\cdot 0000178$. The required angle of course lies between the two angles which we have taken from the Table; let n be the number of seconds in its excess above $44^\circ\ 11'\ 30''$; then

$$\cdot 0000204 : \cdot 0000178 :: 10 : n,$$

therefore $$n = 10 \times \frac{\cdot 0000178}{\cdot 0000204} = \frac{1780}{204} = 8 \cdot 7.$$

Therefore the required angle is $44^{\circ}\, 11'\, 38''\cdot 7$.

175. It will not be necessary to give examples for the other Trigonometrical Functions; the important fact to be remembered is that in the first quadrant the tabular logarithms of the tangent and secant *increase* as the angle increases, and the tabular logarithms of the cotangent and cosecant *decrease* as the angle increases; thus the tangent and secant are treated in the same way as the sine, and the cotangent and cosecant in the same way as the cosine.

EXAMPLES.

1. Given $\log 12440 = 4 \cdot 0948204$,
$\log 12441 = 4 \cdot 0948553$,
find $\log 12440 \cdot 35$.

2. Given $\log 1 \cdot 0686 = \cdot 0288152$,
$\log 1 \cdot 0687 = \cdot 0288558$,
find the number of which the logarithm is $\cdot 0288355$.

3. Given $\log 23456 = 4 \cdot 3702540$,
$\log 23457 = 4 \cdot 3702725$,
form a table of proportional parts for the intermediate numbers, and find $\log \cdot 2345638$.

4. Find the number whose logarithm is $-(1 \cdot 8753145)$, having given

$$\log 1 \cdot 3325 = \cdot 1246672, \quad \log 1 \cdot 3326 = \cdot 1246998.$$

5. Given $\log 3 \cdot 855 = \cdot 5860244$,
$\log 3 \cdot 8551 = \cdot 5860356$,
find $\log (\cdot 00385504)^{\frac{1}{4}}$.

6. Given $\log 24 = 1 \cdot 3802112$,
$\log 4 \cdot 8989 = \cdot 6900986$,
$\log 4 \cdot 8990 = \cdot 6901074$,
find $(24)^{\frac{1}{2}}$ to six places of decimals.

7. Given

$$\log 14271 = 4{\cdot}1544544,$$
$$\log 20313 = 4{\cdot}3077741,$$
$$\log 20314 = 4{\cdot}3077954,$$

find $(142{\cdot}71)^{\frac{1}{7}}$.

8. Given

$$\log 7 = {\cdot}8450980,$$
$$\log 58751 = 4{\cdot}7690153,$$
$$\log 58752 = 4{\cdot}7690227,$$

find $({\cdot}07)^{\frac{1}{3}}$ to seven significant figures.

9. Given $\log 2 = {\cdot}3010300$, $\log 5{\cdot}743491 = {\cdot}7591760$, find the fifth root of ·0625.

10. Given $\log 2{\cdot}7 = {\cdot}4313638$, $\log 5{\cdot}172818 = {\cdot}7137272$, find the value of $27^{-\frac{1}{2}}$.

11. Given $\log 71968 = 4{\cdot}8571394$, diff. for $1 = {\cdot}0000060$, find the value of $\sqrt[5]{({\cdot}0719686)}$ to seven places of decimals.

12. Given $\log 103 = 2{\cdot}0128372$, $\log 7440942 = 6{\cdot}871628$, find $(1{\cdot}03)^{-10}$.

13. Find the value of $64\{1-(1{\cdot}05)^{-20}\}$, having given

$$\log 105 = 2{\cdot}0211893, \quad \log 37689 = 4{\cdot}5762140.$$

14. Find approximately $5^{\sqrt{5}}$, having given

$$\log 2 = {\cdot}301030, \quad \log 1{\cdot}562944 = {\cdot}193943,$$
$$\log 349485 = 5{\cdot}543428, \quad \log 3{\cdot}655 = {\cdot}562887,$$
$$\log 3{\cdot}656 = {\cdot}563006.$$

15. Having given

$\log 12 = 1{\cdot}0791812$, $\log 1{\cdot}257915 = {\cdot}0996512$,
$\log 1{\cdot}121568 = {\cdot}0498256$, find the value of

$$(1{\cdot}44)^{-6} - (1{\cdot}44)^{-12}.$$

16. Having given

$\log 105 = 2{\cdot}0211893$, $\log 5303214 = 6{\cdot}7245391$,
$\log 3768894 = 6{\cdot}576214$, find the value of

$$\frac{1}{{\cdot}05}\left\{\frac{1}{(1{\cdot}05)^{12}} - \frac{1}{(1{\cdot}05)^{20}}\right\}.$$

17. Given $\sin 47° = \cdot 7313537$,
$\sin 48° = \cdot 7431448$,
find $\sin 47° 1'$.

18. Given $\sin 7° 17' = \cdot 1267761$,
$\sin 7° 18' = \cdot 1270646$,
find $\sin 7° 17' 25''$.

19. Given $L \sin 17° 1' = 9 \cdot 4663483$,
$L \sin 17° = 9 \cdot 4659353$,
find $L \sin 17° 0' 12''$.

20. Given $L \sin 26° 24' = 9 \cdot 6480038$,
$L \sin 26° 25' = 9 \cdot 6482582$,
find $L \sin 26° 24' 12''$.

21. Given $L \cot 72° 15' = 9 \cdot 5052891$,
$L \cot 72° 16' = 9 \cdot 5048538$,
find $L \cot 72° 15' 35''$.

22. Given $L \cot 81° 46' = 9 \cdot 1604569$, diff. for $10'' = \cdot 0001486$, find the angle whose $L \cot$ is $9 \cdot 1603493$.

23. Given $L \cos 20° 35' 20'' = 9 \cdot 9713351$, difference for $10'' = \cdot 0000079$, find the angle whose $L \cos$ is $9 \cdot 9713383$.

24. Given $L \cos 34° 24' = 9 \cdot 9165137$, diff. for $1' = \cdot 0000865$, find $L \cos 34° 24' 26''$, and also the angle whose $L \cos$ is $9 \cdot 9165646$.

25. Given $L \sin 37° 19' = 9 \cdot 7826301$, diff. for $1' = \cdot 0001657$,
$L \cos 37° 19' = 9 \cdot 9005294$, diff. for $1' = \cdot 0000963$,
find $L \sec 37° 19' 47''$, and $L \cot 37° 19' 47''$.

26. Given $L \sin 32° 18' = 9 \cdot 7278277$, diff. for $1' = \cdot 0001998$,
$L \cos 32° 18' = 9 \cdot 9269913$, diff. for $1' = \cdot 0000799$,
find L sine, L cosine, and L tangent of $32° 18' 24'' \cdot 6$.

XII. THEORY OF PROPORTIONAL PARTS.

176. We shall now investigate the *principle of proportional parts*, the truth of which was assumed throughout the preceding chapter. The logarithms in the present chapter are supposed to be logarithms to the base 10; and we will suppose that the Table of logarithms is calculated to seven places of decimals, and that it contains the logarithms of every whole number from 1 to 100000.

177. *To shew that the change of the logarithm is approximately proportional to the change of the number.*

We know that $\log(n+d) - \log n = \log\frac{n+d}{n} = \log\left(1+\frac{d}{n}\right)$,

and by Art. 148, $\log\left(1+\frac{d}{n}\right) = \mu\left(\frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \dots\right)$,

where μ is the *modulus*, so that $\mu = \cdot 43429448\dots\dots$

Suppose that n is an integer containing five figures so that n is not less than 10000, and suppose that d is not greater than unity. Then $\frac{\mu d^2}{2n^2}$ is less than $\frac{1}{4}\left(\frac{1}{10000}\right)^2$, and *a fortiori* less than $\cdot 000000003$; $\frac{\mu d^3}{3n^3}$ is less than one ten-thousandth part of this, and so on.

Hence at least as far as *seven* places of decimals we have

$$\log(n+d) - \log n = \frac{\mu d}{n}.$$

This equation establishes the required result; for it shews that if the number be changed from n to $n+d$ the corresponding change in the logarithm is approximately $\frac{\mu d}{n}$, that is, the *change of the logarithm is approximately proportional to the change of the number.*

178. The principle of proportional parts is thus shewn to hold in the case of the logarithms of numbers to a sufficient degree of accuracy for practical use. For when we wish to find the logarithm of a given number we can suppose the decimal point in the

number placed after the fifth figure, so that the number is thus made to lie between two which differ by unity and which are both contained in the Table; and we have shewn that as far as seven places of decimals the change of the logarithm is proportional to the change of the number. Then we can if necessary change the position of the decimal point and make the corresponding change in the *characteristic* of the logarithm; and thus we finally obtain the logarithm of the original given number. Similarly we may proceed if we want to find the number which corresponds to a given logarithm lying between two in the Table.

179. We will now shew how the result of Art. 177 is applied in practice. We have

$$\log(n+d) - \log n = \frac{\mu d}{n},$$

also $$\log(n+1) - \log n = \frac{\mu}{n} = \delta \text{ suppose},$$

thus $$\log(n+d) = \log n + d\delta.$$

Now δ being the difference of two known logarithms is furnished immediately by the Table; and to obtain the logarithm of $(n+d)$ we multiply this known quantity δ by the given fraction d and add the product to the logarithm of n. This is the rule which was used in the preceding chapter, Art. 153, in order to find the logarithm of a given number.

Again, suppose we require the number which corresponds to a given logarithm. Let n and $n+1$ be integers between which the required number lies, and denote the required number by $n+d$. Then $\log(n+d) - \log n$ is known; call it x, and let δ denote the known quantity $\log(n+1) - \log n$; thus $d\delta = x$; therefore $d = \frac{x}{\delta}$. This is the rule which was used in the preceding chapter, Art. 157.

180. We shall now proceed to examine how far the principle of proportional parts holds in the case of the natural Trigonometrical Functions; this we shall do by considering these Functions sepa-

rately. We shall suppose throughout this chapter that the angles which occur are *positive* angles *not exceeding a right angle;* this is sufficient because it has been shewn that any Trigonometrical Function of *any* angle is equal to the same Function of some positive angle not exceeding a right angle; see Art. 55.

181. *To prove that in general the change of the sine of an angle is approximately proportional to the change of the angle.*

We have $\sin(\theta+h)-\sin\theta=\sin h\cos\theta-\sin\theta(1-\cos h)$

$$=\sin h\cos\theta\left(1-\tan\theta\frac{1-\cos h}{\sin h}\right)$$

$$=\sin h\cos\theta\left(1-\tan\theta\tan\frac{h}{2}\right).$$

Let us now suppose that h is the circular measure of a very small angle so that $\sin h=h$ approximately; thus, approximately,

$$\sin(\theta+h)-\sin\theta=h\cos\theta\left(1-\tan\theta\tan\frac{h}{2}\right);$$

let us also suppose that θ is not very nearly equal to $\frac{\pi}{2}$ so that $\tan\theta$ is not very large, and thus $\tan\theta\tan\frac{h}{2}$ may be neglected. We have then, approximately,

$$\sin(\theta+h)-\sin\theta=h\cos\theta,$$

and this establishes the proposition.

Similarly, $\sin(\theta-h)-\sin\theta=-h\cos\theta$ approximately.

182. We may however require to know more exactly the amount of error to which we are liable in using the result of the preceding article; this point we will now examine. The approximate value of $\sin(\theta+h)-\sin\theta$, is $h\cos\theta$, while the exact value is $\sin h\cos\theta-(1-\cos h)\sin\theta$; thus to obtain the approximate value we change $\sin h$ into h in the first term of the exact value, and we neglect the second term of the exact value. First then consider the error produced by writing h for $\sin h$. The circular measure of an angle

of one degree is $\frac{\pi}{180}$; and by Art. 130 $\sin h$ cannot differ from h by so much as $\frac{h^3}{6}$, so that it may be shewn that for an angle of one degree the sine cannot differ from the circular measure by so much as ·0000001. Hence if our calculations extend to only seven places of decimals no error will be introduced by changing $\sin h$ into h even for an angle of *one degree*, and *a fortiori* no error will be introduced by the change if we restrict h to be not greater than the circular measure of an angle of *one minute*. Next consider the error produced by neglecting the term $\sin\theta(1-\cos h)$, that is, $2\sin\theta\sin^2\frac{h}{2}$. Since $\sin\theta$ is never greater than unity and $\sin\frac{h}{2}$ is less than $\frac{h}{2}$, the value of the term neglected is less than $\frac{h^2}{2}$; and if h be the circular measure of an angle of one minute $\frac{h^2}{2}$ is less than ·0000001. Hence if our calculations extend to only seven places of decimals no error will be introduced by neglecting the term $\sin\theta(1-\cos h)$ if we restrict h to be not greater than the circular measure of an angle of one minute.

Therefore if we have a Table of natural sines calculated for every minute to seven places of decimals, no error will be introduced by our calculating to seven places of decimals the sine of an angle which lies between two in the Table from the formula

$$\sin(\theta+h)-\sin\theta=h\cos\theta.$$

183. We will now shew how this result is applied in practice. Suppose that we have a Table of natural sines calculated for every minute, and that we require the sine of an angle which lies between two in the Table. Let k be the circular measure of an angle of one minute; let θ and $\theta+k$ be the circular measures of the angles in the Table between which the given angle lies, and let $\theta+h$ be the circular measure of the given angle. Then

$$\sin(\theta+k)-\sin\theta=k\cos\theta=\delta \text{ suppose,}$$

$$\sin(\theta+h)-\sin\theta = h\cos\theta = \frac{h}{k}\delta;$$

thus $$\sin(\theta+h) = \sin\theta + \frac{h}{k}\delta = \sin\theta + \frac{s}{60}\delta,$$

where s is the number of seconds in the angle of which h is the circular measure. Now δ is the difference between two consecutive sines in the Table, and is therefore furnished immediately by the Table, and we must multiply this known quantity by $\frac{s}{60}$ and add the result to $\sin\theta$ in order to obtain $\sin(\theta+h)$. This is the rule which was used in the preceding chapter, Art. 163.

Again suppose that we require the angle which corresponds to a given natural sine. Let k be the circular measure of an angle of one minute; θ and $\theta+k$ the circular measures of angles in the Table between which the required angle must lie, and let $\theta+h$ be the circular measure of the required angle. Then $\sin(\theta+h)-\sin\theta$ is known; call it x, and let δ denote the known quantity $\sin(\theta+k)-\sin\theta$; therefore $\frac{h\delta}{k}=x$, therefore $\frac{h}{k}=\frac{x}{\delta}$; let s be the number of seconds in the angle of which the circular measure is h, then $\frac{s}{60}=\frac{x}{\delta}$, therefore $s=\frac{60x}{\delta}$. This is the rule which was used in the preceding chapter, Art. 164.

184. When θ is nearly $\frac{\pi}{2}$, since $\cos\theta$ is then very small, the term $h\cos\theta$ will be very small if h be the circular measure of a small angle. Thus the difference between the natural sines of two angles, each of which is nearly equal to a right angle, is very small; this is expressed by saying that the differences in the sines of consecutive angles are nearly *insensible* when the angles are nearly equal to a right angle. There is also another point to be noticed in this case; we have

$$\sin(\theta+h)-\sin\theta = \sin h\cos\theta - (1-\cos h)\sin\theta;$$

the *ratio* of the second term to the first is numerically

$$\frac{\sin\theta\,(1-\cos h)}{\cos\theta\sin h},$$

that is, $\tan\theta\tan\frac{h}{2}$, and when θ is nearly equal to $\frac{\pi}{2}$ this *ratio* will be a sensible quantity unless $\frac{h}{2}$ be extremely small. Thus the second term ought not to be rejected in *comparison with the first term* unless $\frac{h}{2}$ be extremely small. This is expressed by saying that the differences in the sines of consecutive angles are *irregular* when the angles are nearly equal to a right angle. In the present case this *irregularity* is not of much importance on account of the accompanying *insensibility.*

185. We have shewn that, approximately,

$$\sin(\theta+h)-\sin\theta=h\cos\theta;$$

change θ into $\frac{\pi}{2}-\theta'$, thus

$$\sin\left(\frac{\pi}{2}-\theta'+h\right)-\sin\left(\frac{\pi}{2}-\theta'\right)=h\cos\left(\frac{\pi}{2}-\theta'\right),$$

that is,
$$\cos(\theta'-h)-\cos\theta'=h\sin\theta';$$

and by changing the sign of h

$$\cos(\theta'+h)-\cos\theta'=-h\sin\theta'.$$

It is convenient to deduce this formula from that already proved, because we thus know, without a new investigation, the amount of error to which we are liable in using it; it may however be proved independently, as we will now shew.

186. *To prove that in general the change of the cosine of an angle is approximately proportional to the change of the angle.*

We have

$$\cos(\theta-h)-\cos\theta=\sin h\sin\theta-\cos\theta\,(1-\cos h)$$

$$= \sin h \sin \theta \left(1 - \cot \theta \frac{1 - \cos h}{\sin h}\right)$$

$$= \sin h \sin \theta \left(1 - \cot \theta \tan \frac{h}{2}\right).$$

Let us now suppose that h is the circular measure of a very small angle, so that $\sin h = h$ approximately; thus, approximately,

$$\cos (\theta - h) - \cos \theta = h \sin \theta \left(1 - \cot \theta \tan \frac{h}{2}\right);$$

let us also suppose that θ is not very small, so that $\cot \theta$ is not very large, and thus $\cot \theta \tan \frac{h}{2}$ may be neglected. We have then, approximately,

$$\cos (\theta - h) - \cos \theta = h \sin \theta,$$

and by changing the sign of h,

$$\cos (\theta + h) - \cos \theta = - h \sin \theta;$$

and this establishes the proposition.

187. From the result of the preceding article, we can deduce the rule used in Arts. 165, 166 of the preceding chapter; the method is the same as that which we have already given in Art. 183. The only peculiarity to notice is that the cosine *diminishes* as the angle *increases.*

And by proceeding as in Art. 184 we see that the differences in the cosines of consecutive angles are nearly *insensible* and are also *irregular* when the angles are very small.

188. *To prove that in general the change of the tangent of an angle is approximately proportional to the change of the angle.*

We have $\tan (\theta + h) - \tan \theta = \frac{\sin (\theta + h)}{\cos (\theta + h)} - \frac{\sin \theta}{\cos \theta}$

$$= \frac{\sin (\theta + h) \cos \theta - \cos (\theta + h) \sin \theta}{\cos (\theta + h) \cos \theta} = \frac{\sin (\theta + h - \theta)}{\cos (\theta + h) \cos \theta} = \frac{\sin h}{\cos (\theta + h) \cos \theta}$$

$$= \frac{\sin h}{\cos^2 \theta (\cos h - \sin h \tan \theta)} = \frac{\tan h}{\cos^2 \theta (1 - \tan \theta \tan h)}.$$

Let us now suppose that h is so small that we may put h for $\tan h$, and also that θ is not nearly equal to $\frac{\pi}{2}$ so that $\tan\theta\tan h$ may be neglected. We have then, approximately,

$$\tan(\theta+h)-\tan\theta=\frac{h}{\cos^2\theta}=h\sec^2\theta,$$

also by changing the sign of h

$$\tan(\theta-h)-\tan\theta=-h\sec^2\theta;$$

this establishes the proposition.

189. From the result of the preceding article we obtain the same rule for the tangent as we obtained in Art. 183 for the sine. We will now proceed to examine the amount of error to which we are liable in using the approximate formula of the preceding article. We have

$$\frac{\tan h}{\cos^2\theta(1-\tan\theta\tan h)}=\tan h\sec^2\theta(1-\tan\theta\tan h)^{-1}$$
$$=\tan h\sec^2\theta(1+\tan\theta\tan h+\tan^2\theta\tan^2 h+\ldots);$$

thus if we take only the first term $\tan h\sec^2\theta$ we neglect a series of terms beginning with $\tan^2 h\sec^2\theta\tan\theta$, that is approximately $h^2(1+\tan^2\theta)\tan\theta$. Now if we have a table of natural tangents calculated for every minute and we wish to find the natural tangents of intermediate angles the greatest value of h is the circular measure of one minute, that is, $\frac{\pi}{180\times 60}$, or $\cdot 0003$ approximately. Hence the numerical value of the greatest error is not less than $(\cdot 0003)^2(1+\tan^2\theta)\tan\theta$, and therefore even if θ be not greater than $\frac{\pi}{4}$ we are liable to an error in the seventh place of decimals. If, however, we have a table calculated for every ten seconds the greatest value of h is the circular measure of ten seconds, that is, $\frac{\pi}{180\times 60\times 6}$, or $\cdot 00005$ approximately; in this case we shall be free from error in the seventh place of decimals until $\tan\theta$ is about as great as 6; the table shews that $\tan 80^\circ$ is rather less than 6.

190. Since $\tan(\theta+h)-\tan\theta = h\sec^2\theta$ approximately, and $\sec\theta$ is never less than unity, the differences of consecutive tangents are never *insensible;* but as we have shewn in the preceding article, the differences are *irregular* when the angles are nearly right angles.

191. We have shewn that approximately

$$\tan(\theta+h)-\tan\theta = h\sec^2\theta;$$

change θ into $\frac{\pi}{2}-\theta'$, thus

$$\tan\left(\frac{\pi}{2}-\theta'+h\right)-\tan\left(\frac{\pi}{2}-\theta'\right) = h\sec^2\left(\frac{\pi}{2}-\theta'\right),$$

that is $$\cot(\theta'-h)-\cot\theta' = h\,\mathrm{cosec}^2\,\theta',$$

and by changing the sign of h

$$\cot(\theta'+h)-\cot\theta' = -h\,\mathrm{cosec}^2\,\theta'.$$

This may be proved independently, as we will now shew.

192. *To prove that in general the change of the cotangent of an angle is approximately proportional to the change of the angle.*

We have $$\cot(\theta-h)-\cot\theta = \frac{\cos(\theta-h)}{\sin(\theta-h)}-\frac{\cos\theta}{\sin\theta}$$

$$= \frac{\cos(\theta-h)\sin\theta-\cos\theta\sin(\theta-h)}{\sin(\theta-h)\sin\theta} = \frac{\sin(\theta-\theta+h)}{\sin(\theta-h)\sin\theta}$$

$$= \frac{\sin h}{\sin(\theta-h)\sin\theta} = \frac{\sin h}{\sin^2\theta(\cos h-\sin h\cot\theta)}$$

$$= \frac{\tan h}{\sin^2\theta(1-\tan h\cot\theta)}.$$

Let us now suppose that h is so small, that we may put h for $\tan h$, and also that θ is not very small, so that $\cot\theta\tan h$ may be neglected. We have then approximately

$$\cot(\theta-h)-\cot\theta = \frac{h}{\sin^2\theta} = h\,\mathrm{cosec}^2\,\theta,$$

also by changing the sign of h

$$\cot(\theta+h)-\cot\theta = -h\,\mathrm{cosec}^2\,\theta;$$

this establishes the proposition.

193. *To prove that in general the change of the secant of an angle is proportional to the change of the angle.*

We have $$\sec(\theta+h)-\sec\theta=\frac{1}{\cos(\theta+h)}-\frac{1}{\cos\theta}$$

$$=\frac{\cos\theta-\cos(\theta+h)}{\cos\theta\cos(\theta+h)}=\frac{\sin h\sin\theta+(1-\cos h)\cos\theta}{\cos^2\theta(\cos h-\sin h\tan\theta)}$$

$$=\frac{\tan h\sin\theta\left(1+\tan\frac{h}{2}\cot\theta\right)}{\cos^2\theta(1-\tan\theta\tan h)}.$$

Let us now suppose that h is so small that we may put h for $\tan h$, and also that θ is neither very small nor very nearly equal to $\frac{\pi}{2}$, so that $\tan\theta\tan h$ and $\cot\theta\tan\frac{h}{2}$ may be neglected. We have then approximately

$$\sec(\theta+h)-\sec\theta=\frac{h\sin\theta}{\cos^2\theta}=h\sin\theta\sec^2\theta,$$

also by changing the sign of h

$$\sec(\theta-h)-\sec\theta=-h\sin\theta\sec^2\theta;$$

this establishes the proposition.

194. We have shewn that approximately

$$\sec(\theta+h)-\sec\theta=h\sin\theta\sec^2\theta;$$

change θ into $\frac{\pi}{2}-\theta'$, thus

$$\sec\left(\frac{\pi}{2}-\theta'+h\right)-\sec\left(\frac{\pi}{2}-\theta'\right)=h\sin\left(\frac{\pi}{2}-\theta'\right)\sec^2\left(\frac{\pi}{2}-\theta'\right),$$

that is $$\operatorname{cosec}(\theta'-h)-\operatorname{cosec}\theta'=h\cos\theta'\operatorname{cosec}^2\theta',$$

and by changing the sign of h

$$\operatorname{cosec}(\theta'+h)-\operatorname{cosec}\theta'=-h\cos\theta'\operatorname{cosec}^2\theta'.$$

This may also be proved independently.

195. The amount of error to which we are liable in using the approximate formulæ of the preceding two articles may be investigated as in Art. 189. It will be seen that the differences of

consecutive secants are *insensible* and *irregular* when the angles are very small, and they are *irregular* when the angles are nearly right angles; the differences of consecutive cosecants are *irregular* when the angles are small, and *insensible* and *irregular* when the angles are nearly right angles.

We will now proceed to examine how far the principle of proportional parts holds in the case of the *Logarithmic Trigonometrical Functions.*

196. *To prove that in general the change of the tabular logarithmic sine of an angle is approximately proportional to the change of the angle.*

We have approximately $\sin(\theta+h)=\sin\theta+h\cos\theta$,

therefore $$\frac{\sin(\theta+h)}{\sin\theta}=1+h\cot\theta;$$

therefore $\log\sin(\theta+h)-\log\sin\theta=\log\dfrac{\sin(\theta+h)}{\sin\theta}=\log(1+h\cot\theta)$,

and $\log(1+h\cot\theta)=\mu h\cot\theta$ approximately (Art. 148), where μ is the *modulus;* thus approximately

$$\log\sin(\theta+h)-\log\sin\theta=\mu h\cot\theta,$$

also by changing the sign of h

$$\log\sin(\theta-h)-\log\sin\theta=-\mu h\cot\theta.$$

If L stand for *tabular* logarithm, we have

$$L\sin(\theta+h)=10+\log\sin(\theta+h),$$
$$L\sin\theta=10+\log\sin\theta;$$

therefore $$L\sin(\theta\pm h)-L\sin\theta=\pm\mu h\cot\theta.$$

This establishes the proposition.

197. We will now shew that in general the principle of proportional parts holds approximately in the case of the other tabular logarithmic functions, and then we will consider the amount of error to which we are liable in using the approximate formulæ.

198. We have shewn that approximately

$$L\sin(\theta+h)-L\sin\theta=\mu h\cot\theta,$$

change θ into $\frac{\pi}{2}-\theta'$, thus

$$L \sin\left(\frac{\pi}{2}-\theta'+h\right)-L \sin\left(\frac{\pi}{2}-\theta'\right)=\mu h \cot\left(\frac{\pi}{2}-\theta'\right),$$

that is $$L \cos(\theta'-h)-L \cos\theta'=\mu h \tan\theta',$$

and by changing the sign of h

$$L \cos(\theta'+h)-L \cos\theta'=-\mu h \tan\theta'.$$

This proves the principle in the case of the tabular logarithmic cosines.

199. We have shewn that approximately

$$\log \sin(\theta+h)-\log \sin\theta=\mu h \cot\theta,$$

and $$\log \cos(\theta+h)-\log \cos\theta=-\mu h \tan\theta;$$

then by subtraction

$$\log \sin(\theta+h)-\log \cos(\theta+h)-\{\log \sin\theta-\log \cos\theta\}=\mu h(\cot\theta+\tan\theta),$$

that is $$\log \tan(\theta+h)-\log \tan\theta=\frac{2\mu h}{\sin 2\theta},$$

therefore $$L \tan(\theta+h)-L \tan\theta=\frac{2\mu h}{\sin 2\theta},$$

and by changing the sign of h

$$L \tan(\theta-h)-L \tan\theta=-\frac{2\mu h}{\sin 2\theta}.$$

This proves the principle in the case of the tabular logarithmic tangents. By changing θ into $\frac{\pi}{2}-\theta'$ we obtain

$$L \cot(\theta'\mp h)-L \cot\theta'=\pm\frac{2\mu h}{\sin 2\theta'};$$

this proves the principle in the case of the tabular logarithmic cotangents.

200. We have shewn that approximately

$$\log \sin(\theta+h)-\log \sin\theta=\mu h \cot\theta,$$

therefore $$\log\frac{1}{\sin(\theta+h)}-\log\frac{1}{\sin\theta}=-\mu h \cot\theta,$$

that is $$\log \operatorname{cosec}(\theta+h)-\log \operatorname{cosec}\theta=-\mu h \cot\theta,$$

therefore $\quad L \operatorname{cosec}(\theta + h) - L \operatorname{cosec}\theta = -\mu h \cot\theta,$

also by changing the sign of h

$$L \operatorname{cosec}(\theta - h) - L \operatorname{cosec}\theta = \mu h \cot\theta;$$

this proves the principle in the case of the tabular logarithmic cosecants. By changing θ into $\frac{\pi}{2} - \theta'$, we obtain

$$L \sec(\theta' \mp h) - L \sec\theta' = \mp \mu h \tan\theta';$$

this proves the principle in the case of the tabular logarithmic secants.

201. From the results of Arts. 196—200 we obtain the rules which were exemplified in Arts. 171—174. It will be observed that we have deduced the approximate formulæ for all the other logarithmic functions from that of the logarithmic sine; thus if we investigate the amount of error to which we are liable in the case of the logarithmic sine, we shall know the amount of error for all the other logarithmic functions. The approximate formulæ however for the other logarithmic functions may be obtained independently, and we will for example give the investigations for the logarithmic cosine and the logarithmic tangent.

202. *To prove that in general the change of the tabular logarithmic cosine of an angle is approximately proportional to the change of the angle.*

We have approximately $\quad \cos(\theta - h) = \cos\theta + h\sin\theta,$

therefore $\quad \dfrac{\cos(\theta - h)}{\cos\theta} = 1 + h\tan\theta,$

therefore $\log\cos(\theta - h) - \log\cos\theta = \log\dfrac{\cos(\theta - h)}{\cos\theta} = \log(1 + h\tan\theta),$

and $\quad \log(1 + h\tan\theta) = \mu h\tan\theta$ approximately (Art. 148),

therefore $\quad \log\cos(\theta - h) - \log\cos\theta = \mu h\tan\theta$ approximately,

therefore $\quad L\cos(\theta - h) - L\cos\theta = \mu h\tan\theta,$

and by changing the sign of h

$$L\cos(\theta + h) - L\cos\theta = -\mu h\tan\theta.$$

203. *To prove that in general the change of the tabular logarithmic tangent of an angle is approximately proportional to the change of the angle.*

We have approximately $\tan(\theta + h) = \tan\theta + h\sec^2\theta$,

therefore $$\frac{\tan(\theta+h)}{\tan\theta} = 1 + \frac{h\sec^2\theta}{\tan\theta} = 1 + 2h\operatorname{cosec}2\theta,$$

therefore $$\log\tan(\theta+h) - \log\tan\theta = \log(1 + 2h\operatorname{cosec}2\theta)$$
$$= 2\mu h\operatorname{cosec}2\theta \text{ approximately,}$$

therefore $$L\tan(\theta+h) - L\tan\theta = 2\mu h\operatorname{cosec}2\theta,$$

and by changing the sign of h,

$$L\tan(\theta-h) - L\tan\theta = -2\mu h\operatorname{cosec}2\theta.$$

204. We will now proceed to consider the amount of error to which we are liable in using the approximate formula

$$L\sin(\theta+h) - L\sin\theta = \mu h\cot\theta.$$

In obtaining this formula $\log(1 + h\cot\theta)$ was taken equal to $\mu h\cot\theta$, so that the square and higher powers of $h\cot\theta$ were neglected. But when θ is very small $\cot\theta$ is very large, and thus $h^2\cot^2\theta$ may be too large to be neglected; this case then will require further examination.

We have shewn in Art. 181 that

$$\sin(\theta+h) - \sin\theta = \sin h\cos\theta\left(1 - \tan\theta\tan\frac{h}{2}\right);$$

let us suppose h so small that we may write h for $\sin h$ and $\frac{h}{2}$ for $\tan\frac{h}{2}$; thus approximately

$$\sin(\theta+h) - \sin\theta = h\cos\theta - \frac{h^2}{2}\sin\theta,$$

therefore $$\frac{\sin(\theta+h)}{\sin\theta} = 1 + h\cot\theta - \frac{h^2}{2},$$

therefore $$\log \frac{\sin(\theta+h)}{\sin\theta} = \log\left(1 + h\cot\theta - \frac{h^2}{2}\right)$$

$$= \mu\left(h\cot\theta - \frac{h^2}{2}\right) - \frac{\mu}{2}\left(h\cot\theta - \frac{h^2}{2}\right)^2 + \ldots \text{(Art. 148)}$$

$$= \mu h\cot\theta - \frac{\mu h^2}{2}(1 + \cot^2\theta) + \ldots;$$

thus if we omit powers of h higher than h^2 we have

$$\log\sin(\theta+h) - \log\sin\theta = \mu h\cot\theta - \frac{\mu h^2}{2}\operatorname{cosec}^2\theta.$$

If our Table is calculated to every ten seconds, then the greatest value of h is the circular measure of ten seconds, that is about $\cdot 00005$; and $\mu = \frac{1}{2}$ approximately. Thus the greatest error to which we are liable is about $\frac{6\operatorname{cosec}^2\theta}{10^{10}}$. This error will become sensible in calculations to seven places of decimals if θ is less than an angle of 5°, for the tables shew that the sine of 5° is less than $\frac{1}{10}$, and so the cosecant of 5° is greater than 10.

Thus we see that the differences of consecutive logarithmic sines are *irregular* when the angles are *very small.*

When θ is very nearly a right angle, $\cot\theta$ is very small while $\operatorname{cosec}^2\theta$ is not very small; thus the above formula for $\log\sin(\theta+h) - \log\sin\theta$ shews that the differences of consecutive logarithmic sines are nearly *insensible* when the angles are nearly equal to a right angle, and that these differences are at the same time *irregular.*

From these results we can immediately infer the corresponding results for the logarithms of the other Trigonometrical functions; they will be found enunciated in Art. 206.

205. It appears from the preceding article, that when an angle is *small* it cannot be accurately determined from its logarithmic sine nor the logarithmic sine from the angle by means of the common tables, because although the differences of consecutive logarithmic sines are then sensible, yet they are *irregular.* To obviate this difficulty three methods have been proposed.

First Method. We may have a Table of logarithmic sines calculated for *every second* for the first few degrees of the quadrant; in this case the greatest value of h is the circular measure of *one second*, and thus $\frac{h^2}{2}\operatorname{cosec}^2\theta$ becomes small enough to be neglected.

Second Method. This is called *Delambre's* Method. A Table is constructed which gives the value of $\log\frac{\sin\theta}{\theta}+L\sin 1''$ for every second for the first few degrees of the quadrant.

Let θ be the circular measure of an angle of n seconds, then

$$\theta = n\sin 1'' \text{ approximately (Art. 123)},$$

therefore $\log\frac{\sin\theta}{\theta}=\log\frac{\sin n''}{n\sin 1''}=\log\sin n''-\log n-\log\sin 1''$,

$$=L\sin n''-\log n-L\sin 1'',$$

therefore $\log n = L\sin n''-\left(\log\frac{\sin\theta}{\theta}+L\sin 1''\right).$

If the angle is known, then the Table gives the value of $\log\frac{\sin\theta}{\theta}+L\sin 1''$, and $\log n$ can be found from a Table of the logarithms of numbers; thus the formula enables us to find $L\sin n''$.

If the value of $L\sin n''$ is given, and we have to find n, we proceed as follows; since $L\sin n''$ is known we can find *approximately* the value of the angle, and then from the Table we get the value of $\log\frac{\sin\theta}{\theta}+L\sin 1''$; then the formula gives us $\log n$, and we can find n by an ordinary table of logarithms of numbers. In this operation we are liable to an error by using an approximate value of $\frac{\sin\theta}{\theta}$ instead of the real value. But it may be inferred from Chap. IX. and will be more fully shewn hereafter, that when θ is small $\frac{\sin\theta}{\theta}$ is very nearly equal to $1-\frac{\theta^2}{6}$, and thus a small error in θ will not produce any sensible error in our calculations, since $\log\frac{\sin\theta}{\theta}$ will vary far less rapidly than θ.

Third Method. This is called *Maskelyne's* Method. It may be used if Tables such as those described in the other methods are not accessible.

It may be inferred from Chap. IX. and will be more fully shewn hereafter, that when θ is very small we have approximately

$$\sin\theta = \theta - \frac{\theta^3}{6}, \quad \cos\theta = 1 - \frac{\theta^2}{2};$$

therefore $$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = \left(1 - \frac{\theta^2}{2}\right)^{\frac{1}{3}} \text{ approximately,}$$

$$= (\cos\theta)^{\frac{1}{3}} \text{ approximately,}$$

therefore $$\log\sin\theta = \log\theta + \tfrac{1}{3}\log\cos\theta \text{ approximately.}$$

This formula gives $\log\sin\theta$ at once if θ be given. If $\log\sin\theta$ be given, we must find an approximate value of θ, and then find $\log\cos\theta$ approximately; then we have

$$\log\theta = \log\sin\theta - \tfrac{1}{3}\log\cos\theta.$$

Here since $\cos\theta$ varies far less rapidly than θ, we are free from sensible error by using an *approximate* value of $\log\cos\theta$ instead of the real value. A similar formula may be found for the tangent of a small angle; for

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \left(\theta - \frac{\theta^3}{6}\right)\left(1 - \frac{\theta^2}{2}\right)^{-1} \text{ approximately,}$$

therefore $$\frac{\tan\theta}{\theta} = \left(1 - \frac{\theta^2}{6}\right)\left(1 + \frac{\theta^2}{2}\right)$$

$$= 1 + \frac{\theta^2}{3} = \left(1 - \frac{\theta^2}{2}\right)^{-\frac{2}{3}} \text{ approximately,}$$

therefore $$\log\tan\theta = \log\theta - \tfrac{2}{3}\log\cos\theta \text{ approximately.}$$

206. We will now sum up the results of the investigations of the present chapter.

The principle of proportional parts is applicable to all the trigonometrical functions natural and logarithmic with certain exceptions, which occur when the angles are small or nearly equal to a right angle. In the exceptional cases the differences of consecutive functions are sometimes *irregular only;* sometimes they are nearly *insensible,* and then they are *also irregular.*

For the *natural functions* we have the following exceptional cases. For the sine the differences are insensible when the angles are nearly right angles; for the cosine they are insensible when the angles are small. For the tangent the differences are irregular when the angles are nearly right angles; for the cotangent they are irregular when the angles are small. For the secant the differences are insensible when the angles are small, and irregular when they are nearly right angles; for the cosecant the differences are irregular when the angles are small, and insensible when they are nearly right angles.

For *every logarithmic function* the principle of proportional parts fails both when the angles are small and when they are nearly right angles. For the log sine and the log cosecant the differences are irregular when the angles are small, and insensible when they are nearly right angles. For the log cosine and the log secant the differences are insensible when the angles are small, and irregular when they are nearly right angles. For the log tangent and the log cotangent the differences are irregular when the angles are small and when they are nearly right angles.

207. In using Trigonometrical Tables it is necessary to avoid as much as possible the cases in which the principle of proportional parts does not hold. In other words, we must endeavour to use a Table such that the differences of the function corresponding to given small differences of the angle are both *sensible* and *regular.* If the differences of the function are *insensible* for a certain number of decimal places we cannot by any method determine the value of the function for any intermediate angle, or perform the converse operation, so long as we are restricted to the certain number of decimal places. If the differences of the function are *irregular* we cannot determine the value of the function for an intermediate angle, or perform the converse

operation, by the principle of proportional parts, though we may by retaining the terms which were neglected in the first approximation.

208. If we have to determine an angle from its natural sine or cosine it will be advisable to employ the natural sine if the angle be less than 45°, and the natural cosine if the angle be greater than 45°. For the differences of consecutive sines vary approximately as the cosine of the angle, and the differences of consecutive cosines vary approximately as the sine of the angle; thus the differences of consecutive sines are greater or less than the differences of consecutive cosines according as the angle is less or greater than 45°. A similar remark holds for the logarithmic sine and cosine.

209. The student who is acquainted with the elements of the Differential Calculus will see that all the results of the present chapter may be obtained from Taylor's Theorem; and thus these results may be easily retained in the memory, or at least readily recovered when required. For example, consider the natural sine; we have by Taylor's Theorem

$$\sin(\theta + h) = \sin\theta + h\cos\theta - \frac{h^2}{2}\sin(\theta + \lambda h),$$

where λ is some proper fraction. This formula shews that if we put

$$\sin(\theta + h) = \sin\theta + h\cos\theta$$

the error is less than $\frac{h^2}{2}$. Moreover we see that when θ is small the principle of proportional parts is especially applicable, for then the term $\frac{h^2}{2}\sin(\theta + \lambda h)$ is extremely small in comparison with $h\cos\theta$; and on the other hand, when θ is nearly $\frac{\pi}{2}$ the principle is not so appropriate, because then $\frac{h^2}{2}\sin(\theta + \lambda h)$ may be sensible in comparison with $h\cos\theta$.

Again, by Taylor's Theorem, we have

$$\log \sin (\theta + h) = \log \sin \theta + \mu h \cot \theta - \frac{\mu h^2}{2} \operatorname{cosec}^2 (\theta + \lambda h),$$

where μ is the modulus and λ some proper fraction. This equation shews that the principle of proportional parts is in general applicable for the logarithmic sine, but that the differences of consecutive logarithmic sines are irregular when the angles are small, and insensible and irregular when the angles are nearly right angles.

210. The following application of Taylor's Theorem will give a good mode of estimating the amount of error involved in the principle of proportional parts. Take the logarithmic sine for example; we have

$$\log \sin (\theta + h) = \log \sin \theta + \mu h \cot (\theta + \lambda h),$$

where λ is some proper fraction. Thus the approximation uses $\cot \theta$ instead of $\cot (\theta + \lambda h)$. The true value in fact of $\log \sin (\theta + h) - \log \sin \theta$ must lie between $\mu h \cot \theta$ and $\mu h \cot (\theta + h)$, so that the error is less than $\mu h \{\cot \theta - \cot (\theta + h)\}$.

MISCELLANEOUS EXAMPLES.

1. From one of the angles of a rectangle a perpendicular is drawn to its diagonal, and from the point of their intersection lines are drawn perpendicular to the sides which contain the opposite angle; shew that if p and p' be the lengths of the perpendiculars last drawn, and c the diagonal of the rectangle,

$$p^{\frac{2}{3}} + p'^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

2. If two circles whose radii are a and b touch each other externally, and if θ be the angle contained by the two common tangents to these circles, shew that

$$\sin \theta = \frac{4 (a - b) \sqrt{(ab)}}{(a + b)^2}.$$

3. Given $\sec \alpha \sec \theta + \tan \alpha \tan \theta = \sec \beta$, find $\tan \theta$.

4. Find the limit when $\theta = 0$ of

$$\frac{\sin \frac{\theta}{2} \cos 2\theta}{\text{vers}\, \theta \cot \theta}, \quad \text{and of } \frac{\tan^2 \theta}{\sec 2\theta - 1}.$$

5. Shew that $\cot \frac{\theta}{2}$ is greater than $1 + \cot \theta$ for all values of θ between 0 and π.

6. If $\tan \frac{\theta}{2} = \frac{\tan \theta + c - 1}{\tan \theta + c + 1}$, find $\tan \frac{\theta}{2}$.

7. Find the condition necessary that the same value of θ may satisfy both the equations

$$a \sec^2 \theta - b \cos \theta = 2a, \quad b \cos^2 \theta - a \sec \theta = 2b.$$

8. Eliminate α and β from the equations

$$a = \sin \alpha \cos \beta \sin \theta + \cos \alpha \cos \theta,$$
$$b = \sin \alpha \cos \beta \cos \theta - \cos \alpha \sin \theta,$$
$$c = \sin \alpha \sin \beta \sin \theta.$$

9. Eliminate α and β from the equations

$$b + c \cos \alpha = u \cos (\alpha - \theta), \quad b + c \cos \beta = u \cos (\beta - \theta), \quad \alpha - \beta = 2\delta$$

and shew that $u^2 - 2uc \cos \theta + c^2 = b^2 \sec^2 \delta$.

10. Eliminate x from the equations

$$\frac{a \tan^2 \theta - x}{\tan 2\alpha \tan 2\alpha'} = \frac{2a \tan \theta}{\tan 2\alpha + \tan 2\alpha'} = a - x;$$

and shew that $\theta = \alpha + \alpha'$, or $\frac{\pi}{2} + \alpha + \alpha'$.

11. Eliminate θ and ϕ from the equations

$$\sin \theta + \sin \phi = a, \quad \cos \theta + \cos \phi = b, \quad \cos (\theta - \phi) = c.$$

12. Eliminate θ and ϕ from the equations

$$x \cos \theta + y \sin \theta = a, \quad x \cos (\theta + 2\phi) - y \sin (\theta + 2\phi) = a,$$
$$b \sin (\theta + \phi) = a \sin \phi.$$

13. Eliminate x and y from the equations

$$\tan x + \tan y = a, \quad \cot x + \cot y = b, \quad x + y = c.$$

14. Eliminate θ from the equations

$$\frac{x}{a} = \frac{\sec^2\theta - \cos^2\theta}{\sec^2\theta + \cos^2\theta}, \quad \frac{2b}{y} = \sec^2\theta + \cos^2\theta.$$

15. Eliminate θ from the equations

$$(a+b)\tan(\theta - \phi) = (a-b)\tan(\theta + \phi),$$

$$\cos 2\phi + b\cos 2\theta = c.$$

16. Given $$\frac{x^2}{a^2}\cos\theta = \frac{y^2}{a^2}\cos\theta + \frac{z^2}{b^2}\cos\theta',$$

and $$\frac{x}{\sin(\theta + \theta')} = \frac{y}{\sin(\theta - \theta')} = \frac{z}{\sin 2\theta},$$

shew that $$\frac{\sin\theta}{\sin\theta'} = \frac{b^2}{a^2}.$$

17. Eliminate ϕ from the equations

$$y\cos\phi - x\sin\phi = a\cos 2\phi, \quad y\sin\phi + x\cos\phi = 2a\sin 2\phi;$$

and shew that $$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

18. Eliminate θ and ϕ from the equations

$$\cos\theta = \frac{\sin\beta}{\sin\alpha}, \quad \cos\phi = \frac{\sin\gamma}{\sin\alpha},$$

$$\cos(\theta - \phi) = \sin\beta\sin\gamma;$$

and shew that $$\tan^2\alpha = \tan^2\beta + \tan^2\gamma.$$

19. Eliminate θ from the equations

$$m = \operatorname{cosec}\theta - \sin\theta, \quad n = \sec\theta - \cos\theta.$$

20. Eliminate θ from the equations

$$x \sin \theta - y \cos \theta = \sqrt{(x^2 + y^2)}, \quad \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{x^2 + y^2}.$$

21. Eliminate θ and θ' from the equations

$$a \sin^2 \theta + a' \cos^2 \theta = b, \quad a' \sin^2 \theta' + a \cos^2 \theta' = b',$$

$$a \tan \theta = a' \tan \theta',$$

and shew that $$\frac{1}{b} + \frac{1}{b'} = \frac{1}{a} + \frac{1}{a'}.$$

22. Given $x^2 + y^2 = a^2 + b^2, \quad xy = ab \sin \alpha,$

$$\frac{\cos^2 \theta}{x^2} + \frac{\sin^2 \theta}{y^2} = \frac{1}{a^2},$$

shew that $$\pm \cot 2\theta = \cot 2\alpha + \frac{a^2}{b^2} \operatorname{cosec} 2\alpha.$$

23. If $\dfrac{\cos x}{a_1} = \dfrac{\cos 2x}{a_2} = \dfrac{\cos 3x}{a_3}$, shew that

$$\sin^2 \frac{x}{2} = \frac{2a_2 - a_1 - a_3}{4a_2}.$$

24. If $$\frac{\sin x}{a_1} = \frac{\sin 3x}{a_3} = \frac{\sin 5x}{a_5},$$

shew that $$\frac{a_1 - 2a_3 + a_5}{a_3} = \frac{a_3 - 3a_1}{a_1}.$$

25. Given $$\frac{\cos x}{a_1} = \frac{\cos (x + \theta)}{a_2} = \frac{\cos (x + 2\theta)}{a_3} = \frac{\cos (x + 3\theta)}{a_4},$$

shew that $$\frac{a_1 + a_3}{a_2} = \frac{a_2 + a_4}{a_3}.$$

26. If $$\sin^2\phi = \frac{\cos 2\alpha \cos 2\alpha'}{\cos^2(\alpha+\alpha')},$$ then

$$\tan^2\frac{\phi}{2} = \frac{\tan\left(\frac{\pi}{4} \pm \alpha\right)}{\tan\left(\frac{\pi}{4} \pm \alpha'\right)}.$$

27. If $$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\phi-\alpha)\cos\beta} + \frac{\cos(\alpha+\theta)\sin\beta}{\cos(\phi-\beta)\sin\alpha} = 0,$$

and $$\frac{\tan\theta\tan\alpha}{\tan\phi\tan\beta} + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = 0,$$

shew that $\tan\theta = \frac{1}{2}(\tan\beta + \cot\alpha)$, $\tan\phi = \frac{1}{2}(\tan\alpha - \cot\beta)$.

28. If $$\frac{2}{1+x} = \frac{\sin\beta\sin\theta}{\cos(\beta-\theta)} = \frac{\tan(\theta-\alpha)}{\cot\beta},$$

prove that $$x^2 = \left(\cot\frac{\alpha}{2} - 2\cot\beta\right)\left(\tan\frac{\alpha}{2} + 2\cot\beta\right).$$

29. Given $\sin\theta\sin\phi = \sin\alpha\sin\beta$, $\tan\phi\cos\beta = \cot\frac{\alpha}{2}$, prove that one of the values of $\sin\frac{\theta}{2}$ is $\sin\frac{\alpha}{2}\sin\beta$.

30. Given $\sin\phi = n\sin\theta$, $\tan\phi = 2\tan\theta$, find the limiting values of n that these equations may coexist.

31. Shew by means of a Trigonometrical formula that

if $x + y + z = xyz$,

then $$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

32. Find the values of v, x, y, z from the equations

$$v = \frac{\sin x}{\sin a} = \frac{\sin y}{\sin b} = \frac{\sin z}{\sin c}; \quad x + y + z = 2\pi.$$

33. Find the limit of $(\cos ax)^{\operatorname{cosec}^2\beta x}$ when x is zero.

34. From a table of natural tangents which goes to 7 places of decimals, shew that an angle may be determined within about $\frac{1}{200}$th part of a second when the angle is nearly 60°.

35. When an angle is very nearly equal to 64° 36′, shew that the angle can be determined from its *L* sine within about $\frac{1}{10}$th of a second; having given $\log_e 10 \,.\, \tan 64° 36' = 4{\cdot}8492$, and the tables going to 7 places of decimals.

36. Shew that

$$\left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 - \tan^2\frac{\alpha}{2^2}\right)\left(1 - \tan^2\frac{\alpha}{2^3}\right)\ldots\ldots ad\ inf. = \frac{\alpha}{\tan\alpha}.$$

37. If A, B, C, be positive angles which satisfy the equation

$$\sin^2 A + \sin^2 B + \sin^2 C = 1,$$

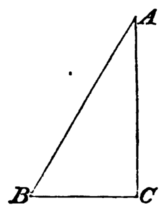
prove that $A + B + C$ is greater than 90°.

38. A circle is drawn touching the tangent and secant of a given angle α, as well as the corresponding arc; find its radius and explain the double value. If one value be equal to the radius of the original circle, shew that $\alpha = \frac{\pi}{3}$.

XIII. RELATIONS BETWEEN THE SIDES OF A TRIANGLE AND THE TRIGONOMETRICAL FUNCTIONS OF THE ANGLES.

211. We shall now investigate certain relations which hold between the sides of a triangle and the Trigonometrical Functions of its angles; these relations will be applied in the following chapter to the *solution of Triangles*. We shall denote the angles of a triangle by the letters A, B, C, and the lengths of the sides respectively opposite to these angles by the letters a, b, c; thus a, b, c are *numbers* expressing the lengths of the sides in terms of some unit of length such as a foot, or a yard, or a mile. The unit of length may be whatever we please, but must be the same for all the sides.

212. *In a right-angled triangle each side is equal to the product of the hypothenuse into the cosine of the adjacent angle.*



Let ABC be a triangle having a right angle at C; then

$$\frac{AC}{AB} = \cos A, \quad \frac{BC}{AB} = \cos B;$$

therefore $b = c\cos A$, $a = c\cos B$.

Since $\cos A = \sin B$ and $\cos B = \sin A$, we may also enunciate the proposition thus—*in a right-angled triangle each side is equal to the product of the hypothenuse into the sine of the opposite angle.*

213. *In any right-angled triangle each side is equal to the product of the tangent of the opposite angle into the other side.*

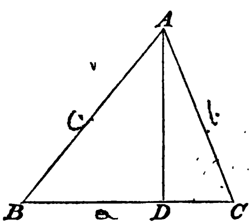
From the figure of the preceding article we have

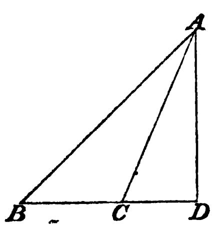
$$\tan A = \frac{BC}{AC}, \quad \tan B = \frac{AC}{BC};$$

therefore $$a = b\tan A, \quad b = a\tan B.$$

Since $\tan A = \cot B$ and $\tan B = \cot A$, we may also enunciate the proposition thus—*in any right-angled triangle each side is equal to the product of the cotangent of the adjacent angle into the other side.*

214. *In any triangle the sides are proportional to the sines of the opposite angles.*





Let ABC be any triangle, and from A draw AD perpendicular to the opposite side meeting that side, or that side produced, in D. If B and C are *acute* angles we have from the left-hand figure,

$$AD = AB \sin B, \text{ and } AD = AC \sin C;$$

therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be *obtuse* we have from the right-hand figure,

$$AD = AB \sin B, \text{ and } AD = AC \sin(180^\circ - C) = AC \sin C;$$

therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be a *right angle*, we have from the figure of Art. 212,

$$AC = AB \sin B,$$

therefore $$\frac{c}{b} = \frac{1}{\sin B} = \frac{\sin C}{\sin B}.$$

Thus it is proved that in every case $\frac{c}{b} = \frac{\sin C}{\sin B}$.

Similarly $\frac{a}{b} = \frac{\sin A}{\sin B}$ and $\frac{a}{c} = \frac{\sin A}{\sin C}$.

The results may be written symmetrically thus,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

215. *To express the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle, and suppose C an *acute* angle. (See the left-hand figure of the preceding article.) Then by Euclid II. 13,

$$AB^2 = BC^2 + AC^2 - 2BC \cdot CD,$$

and $$CD = AC \cos C;$$

therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Next suppose C an *obtuse* angle. (See the right-hand figure of the preceding article.) Then by Euclid II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC \,.\, CD,$$

and $$CD = AC \cos(180° - C) = -AC \cos C,$$

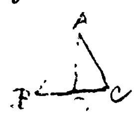
therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Thus in both cases we have $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Moreover when C is a right angle, $a^2 + b^2 = c^2$ and $\cos C$ is zero; thus the formula just found for $\cos C$ is true whatever the angle C may be.

Similarly, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

216. *In every triangle each side is equal to the sum of the product of each of the others into the cosine of the angle which it makes with the first side.*



From the left-hand figure in Art. 214, we have

$$BC = BD + DC = AB \cos B + AC \cos C,$$

that is, $$a = c \cos B + b \cos C.$$

From the right-hand figure in Art. 214, we have

$$BC = BD - DC = AB \cos B - AC \cos(180° - C)$$
$$= AB \cos B + AC \cos C,$$

that is, $$a = c \cos B + b \cos C.$$

Similarly, in every case, we shall have

$$b = a \cos C + c \cos A,$$

and $$c = b \cos A + a \cos B.$$

217. *To express the sine, cosine, and tangent of half an angle of a triangle in terms of the sides.*

We have by Art. 215,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

therefore $$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc};$$

therefore $$\sin^2 \frac{A}{2} = \frac{(a+b-c)(a+c-b)}{4bc}.$$

Let $2s = a + b + c$ so that s is half the sum of the sides of the triangle; then

$$a + b - c = a + b + c - 2c = 2(s - c),$$

$$a + c - b = a + b + c - 2b = 2(s - b).$$

Therefore $$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc},$$

and $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Also $$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc};$$

therefore $$\cos^2 \frac{A}{2} = \frac{(a+b+c)(b+c-a)}{4bc} = \frac{s(s-a)}{bc},$$

and $$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

From the values of $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ we deduce

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The positive sign must be given to the radicals which occur in this article, because $\frac{A}{2}$ is less than a right angle, and therefore its sine, cosine, and tangent are all positive.

Similar expressions hold for the sine, cosine, and tangent of half of each of the other angles.

218. Since $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$, we obtain

$$\sin A = 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$
$$= \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

Or we may find $\sin A$ directly from the known value of $\cos A$;

thus $$\sin^2 A = 1 - \frac{(b^2+c^2-a^2)^2}{4b^2c^2}$$
$$= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4b^2c^2};$$

therefore $$\sin A = \frac{\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}}{2bc};$$

the former expression may be shewn to agree with this by forming the product of the factors s, $s-a$, $s-b$, and $s-c$.

219. We have proved the formulæ in Arts. 214—216 independently from the figures; we may however observe that it is easy to deduce those in any two of the articles from those in the third. Thus we may first establish as in Art. 216, that

$$a = b \cos C + c \cos B,\ b = c \cos A + a \cos C,\ c = a \cos B + b \cos A;$$

multiply the first of these equations by a, the second by b, and the third by c; then add the first two resulting equations and subtract the third; thus we obtain

$$a^2 + b^2 - c^2 = 2ab \cos C.$$

Similarly the other two formulæ of Art. 215 may be deduced.

Then from these results we may proceed as in Arts. 217, 218,

and shew that $$\frac{\sin A}{a} = \frac{2}{abc}\sqrt{s(s-a)(s-b)(s-c)},$$

and that $\frac{\sin B}{b}$ and $\frac{\sin C}{c}$ are equal to the same expression.

Thus $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Or we may begin by establishing the formulæ of Art. 214 directly from the figure, and then proceed as follows,

$$\sin A = \sin(180^\circ - A) = \sin(B + C) = \sin B \cos C + \cos B \sin C;$$

therefore $$1 = \cos C \frac{\sin B}{\sin A} + \cos B \frac{\sin C}{\sin A},$$

$$= \frac{b}{a} \cos C + \frac{c}{a} \cos B;$$

therefore $$a = b \cos C + c \cos B.$$

Similarly the other two formulæ of Art. 216 may be deduced; and then those of Art. 217 will follow in the manner shewn in the beginning of the present article.

220. The reason why an ambiguity of sign occurs in the formulæ for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ of Art. 217 may be explained as on former occasions. It will be observed that we have an expression for $\cos A$, and we proceed to deduce expressions for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$; and in Art. 96 it has been shewn that in this case we may expect two values differing only in sign for each of the required quantities.

221. Since the formulæ in Art. 217 have been strictly demonstrated, they must of course always furnish real values for $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, and $\tan \frac{A}{2}$, if the triangle really exist. That they do so may be easily verified from a known property of a triangle.

Take for example the formula

$$\sin^2 \frac{A}{2} = \frac{(a + b - c)(a + c - b)}{4bc};$$

that this may give a possible value for $\sin\frac{A}{2}$ the expression on the right hand must be *positive* and less than unity. It *is* positive, because from the fact that two sides of a triangle are greater than the third, we have $a+b-c$ positive and $a+c-b$ positive. And the numerator is $a^2-(c-b)^2$, and this is less than the denominator provided a^2 be less than $(c-b)^2+4bc$, that is provided a^2 be less than $(b+c)^2$, which is obviously the case.

MISCELLANEOUS EXAMPLES.

1. The sides of a triangle are x^2+x+1, $2x+1$, and x^2-1; shew that the greatest angle is 120°.

2. If $\cos B = \frac{\sin A}{2\sin C}$, shew that the triangle is isosceles.

3. In a right-angled triangle of which C is the right angle,

$$\cot\frac{A}{2} = \frac{b+c}{a}.$$

4. If $a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$ shew that $\frac{a}{b} = \frac{\cos A}{\cos B}$.

5. The angles of a plane triangle form a geometrical progression of which the common ratio is $\frac{1}{2}$; shew that the greatest side is to the perimeter as $2\sin\frac{\pi}{14}$ to unity.

6. If A', B', C' are the *external* angles of a triangle, shew that

$$2bc\operatorname{vers} A' + 2ca\operatorname{vers} B' + 2ab\operatorname{vers} C' = (a+b+c)^2.$$

7. From the angle A of any triangle ABC a perpendicular AD is drawn upon the base, and from D perpendiculars DE, DF are drawn upon AB, AC respectively; shew that

$$AE \,.\, EB \,.\, \cos^2 C = AF \,.\, FC \,.\, \cos^2 B.$$

8. If a, b, c, be the sides of a triangle and the opposite angles be 2θ, 3θ, 4θ, shew that $\tan^2\theta = \left(\frac{2b}{a+c}\right)^2 - 1$.

9. ABC is a triangle of which C is an obtuse angle; shew that $\tan A \tan B$ is less than unity.

10. If the sides a, b, c of a triangle be in arithmetical progression, shew that

$$\cos\frac{A-C}{2} = 2\sin\frac{B}{2}, \text{ and } a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}.$$

11. If D be the middle point of the side BC of a triangle

$$\cot BAD - \cot B = 2\cot A.$$

12. If an angle of a triangle be divided into two parts such that the sines are in the ratio of the sides adjacent to them respectively, prove that the difference of their cotangents is equal to the difference of the cotangents of the angles opposite to their sides.

13. If the cotangents of the angles of a triangle be in arithmetical progression, the squares of the sides will also be in arithmetical progression.

14. Given the vertical angle and the ratio between the base and altitude of a triangle, find the tangents of the angles into which the vertical angle is divided by the perpendicular drawn from it upon the base.

15. If the base of a triangle be divided into three equal parts, and t_1, t_2, t_3 be the tangents of the angles which they subtend at the vertex

$$\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right).$$

16. If the sines of the angles of a triangle be in arithmetical progression, the product of the tangents of half the greatest and half the least is $\frac{1}{3}$.

17. If the side BC of a triangle be bisected in D and AD be drawn, shew that $\tan ADB = \dfrac{2bc \sin A}{b^2 \sim c^2}$.

18. If A, B, C be the angles of a triangle and $\cot \frac{A}{2}$, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ in arithmetical progression, prove that $\cot \frac{A}{2} \cot \frac{C}{2} = 3$.

19. Straight lines are drawn from the angles A and B of a triangle dividing the angles respectively into parts whose sines are in the ratio of 1 to n; these lines intersect in D; shew that DC either bisects the angle C or divides it into parts whose sines are in the ratio of 1 to n^2.

20. If l be the length of the line which bisects the angle A of a triangle and is terminated by the base, θ the angle which it makes with the base, shew that the perimeter of the triangle

$$= \frac{2l \cos \frac{A}{2} \sin \theta}{\sin \theta - \sin \frac{A}{2}}.$$

21. If θ and ϕ be the greatest and least angles of a triangle the sides of which are in arithmetical progression, prove that

$$4(1 - \cos \theta)(1 - \cos \phi) = \cos \theta + \cos \phi.$$

22. From the angular points of a triangle ABC lines are drawn making each the same angle α towards the same parts with the sides of the triangle taken in order. Shew that these lines will form another triangle similar to the former, and that the linear dimensions of the two triangles are in the ratio of

$$\cos \alpha - \sin \alpha (\cot A + \cot B + \cot C) \text{ to } 1.$$

Shew that in any triangle the relations given in the following examples, from 23 to 40, hold.

23. $a(b \cos C - c \cos B) = b^2 - c^2$.

24. $a(\cos B \cos C + \cos A) = b(\cos A \cos C + \cos B)$

$$= c(\cos A \cos B + \cos C).$$

25. $(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$.

26. $b\cos B+c\cos C=a\cos(B-C)$.

27. $(a+b)\cos C+(b+c)\cos A+(c+a)\cos B=a+b+c$.

28. $(a^2-b^2)\cot C+(b^2-c^2)\cot A+(c^2-a^2)\cot B=0$.

29. $(a-b)\cot\frac{C}{2}+(c-a)\cot\frac{B}{2}+(b-c)\cot\frac{A}{2}=0$.

30. $1-\tan\frac{A}{2}\tan\frac{B}{2}=\frac{2c}{a+b+c}$.

31. $(a+b+c)(\cos A+\cos B+\cos C)$

$$=2a\cos^2\frac{A}{2}+2b\cos^2\frac{B}{2}+2c\cos^2\frac{C}{2}.$$

32. $\frac{\sin^2 A}{a^2}=\frac{\cos A\cos B}{ab}+\frac{\cos A\cos C}{ac}+\frac{\cos B\cos C}{bc}$.

33. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$.

34. $\cos A+\cos B+\cos C=1+\frac{2a\sin B\sin C}{a+b+c}$.

35. $a^2-2ab\cos(60^\circ+C)=c^2-2bc\cos(60^\circ+A)$.

36. $\cot\frac{A}{4}-\operatorname{cosec}\frac{A}{2}:\cot\frac{B}{2}+\cot\frac{C}{2}::b+c-a:2a$.

37. $\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=4\Sigma\left(\Sigma-\cos\frac{A}{2}\right)\left(\Sigma-\cos\frac{B}{2}\right)\left(\Sigma-\cos\frac{C}{2}\right)$,

where $$2\Sigma=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}.$$

38. The perimeter of any triangle is $2c\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}$.

39. If $y\sin^2 A+x\sin^2 B=z\sin^2 B+y\sin^2 C=x\sin^2 C+z\sin^2 A$,

then $$x:y:z::\sin 2A:\sin 2B:\sin 2C.$$

40. $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is less than 1, except when $A=B=C$.

XIV. SOLUTION OF TRIANGLES.

222. In every triangle there are six elements, namely, the three sides and the three angles. The *solution of triangles* is the process by which when the values of a sufficient number of these elements are given we calculate the values of the remaining elements. It will appear as we proceed that when three of the elements are given, the remaining three can be found except when the three angles are given, and then we cannot determine the *lengths* of the sides but only the ratio they bear to each other. We shall have occasion to introduce logarithms into our formulæ, and we shall as before by the word *logarithm* or the abbreviation *log* denote a logarithm to the base 10; and by the letter L placed before any Trigonometrical Function, we shall denote the *tabular logarithm* of that function, which is formed by adding 10 to the logarithm to the base 10.

We shall begin with a right-angled triangle and shall suppose C the right angle.

223. *To solve a right-angled triangle having given the hypothenuse and an acute angle.*

Suppose the hypothenuse and the angle A given; then

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } a = c \sin A,$$

therefore $\log a = \log c + \log \sin A = \log c + L \sin A - 10;$

$$\frac{b}{c} = \sin B, \text{ therefore } b = c \sin B,$$

therefore $\log b = \log c + \log \sin B = \log c + L \sin B - 10;$

Thus B, a, and b are determined.

224. *To solve a right-angled triangle having given the hypothenuse and a side.*

Suppose c and a given; then

$$\sin A = \frac{a}{c}, \ \log \sin A = \log a - \log c;$$

therefore $$L \sin A = 10 + \log a - \log c;$$

this determines A; then $B = 90^\circ - A$.

And $c^2 = a^2 + b^2$, therefore $b^2 = c^2 - a^2 = (c-a)(c+a)$,

therefore $$b = \sqrt{(c-a)(c+a)},$$

$$\log b = \tfrac{1}{2}\log(c-a) + \tfrac{1}{2}\log(c+a).$$

Or we may find b from the formula $b = c \cos A$.

225. *To solve a right-angled triangle having given a side and an acute angle.*

Suppose a and A given; then

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \ \text{therefore } c = \frac{a}{\sin A},$$

$$\log c = \log a - \log \sin A = \log a - L \sin A + 10;$$

$$\frac{a}{b} = \tan A, \ \text{therefore } b = \frac{a}{\tan A},$$

$$\log b = \log a - \log \tan A = \log a - L \tan A + 10.$$

Thus B, c, b are determined.

If a and B are given, then $A = 90^\circ - B$; thus A is known, and we may find c and b as before.

226. *To solve a right-angled triangle having given the two sides.*

Here a and b are given; then

$$\tan A = \frac{a}{b}, \ \text{therefore } \log \tan A = \log a - \log b,$$

therefore $$L \tan A = 10 + \log a - \log b;$$

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \quad \text{therefore } c = \frac{a}{\sin A},$$

therefore $$\log c = \log a - L \sin A + 10.$$

Or we may find c from the formula $c = \sqrt{(a^2 + b^2)}$, but this is not adapted to logarithmic computation.

227. We may remark here that when an angle of a triangle is determined from its cosine, versed sine, tangent, cotangent or secant, no uncertainty can exist about the angle, because only *one* angle exists less than 180° for which any of these functions has an assigned value. But when an angle of a triangle is determined from its sine or cosecant uncertainty *may* exist, since there are two angles less than 180° which have a given sine or a given cosecant. But no uncertainty will exist in the case of a right-angled triangle, because each of the other angles of the triangle must be *acute*.

We now proceed to the solution of oblique-angled triangles.

228. *To solve a triangle having given two angles and a side.*

Suppose A and C the given angles, and b the given side; then $$B = 180^\circ - A - C;$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{therefore } a = \frac{b \sin A}{\sin B},$$

therefore $\log a = \log b + \log \sin A - \log \sin B = \log b + L \sin A - L \sin B$;

similarly $\log c = \log b + L \sin C - L \sin B$.

Thus B, a, and c are determined.

If A and B are the given angles then

$$C = 180^\circ - B - A,$$

and we may proceed as before to find a and c.

229. *To solve a triangle having given two sides and the included angle.*

Suppose b and c the given sides and A the included angle.

We have $$\frac{\sin B}{\sin C} = \frac{b}{c};$$

therefore $$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c},$$

therefore $$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c}, \text{ (Art. 88)},$$

and $$\tan \tfrac{1}{2}(B+C) = \tan \tfrac{1}{2}(180^\circ - A) = \cot \frac{A}{2},$$

therefore $$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{A}{2},$$

therefore $$\log \tan \tfrac{1}{2}(B-C) = \log (b-c) - \log (b+c) + \log \cot \frac{A}{2},$$

therefore $$L \tan \tfrac{1}{2}(B-C) = \log (b-c) - \log (b+c) + L \cot \frac{A}{2};$$

this formula determines $\frac{1}{2}(B-C)$; and $\frac{1}{2}(B+C)$ is known since it is $90^\circ - \frac{A}{2}$; thus B and C can be immediately found.

Also $\frac{a}{c} = \frac{\sin A}{\sin C}$, from which a can be found.

230. In finding a from the expression just quoted we should require three logarithms, namely, those of c, $\sin A$, and $\sin C$; in the following method we shall only require two new logarithms.

We have $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

therefore $$\frac{a}{\sin A} = \frac{b+c}{\sin B + \sin C},$$

and $$\sin B + \sin C = 2 \sin \tfrac{1}{2}(B+C) \cos \tfrac{1}{2}(B-C) \quad \text{(Art. 83)}$$

$$= 2 \cos \frac{A}{2} \cos \tfrac{1}{2}(B-C),$$

therefore $$a = \frac{(b+c)\sin A}{2\cos\frac{A}{2}\cos\frac{1}{2}(B-C)} = \frac{(b+c)\sin\frac{A}{2}}{\cos\frac{1}{2}(B-C)};$$

as the logarithm of $b+c$ has been used in the former part of the solution, we shall only require two new logarithms, namely those of $\sin\frac{A}{2}$ and $\cos\frac{1}{2}(B-C)$.

231. We can also from the given quantities in the preceding article determine the third side *without previously determining the other two angles.* For we have by Art. 215,

$$a^2 = b^2 + c^2 - 2bc\cos A;$$

and we can transform this formula into another, which is adapted to logarithmic computation as follows;

$$a^2 = b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right),$$

$$= (b+c)^2 - 4bc\cos^2\frac{A}{2},$$

$$= (b+c)^2\left\{1 - \frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right\}.$$

Now find an angle θ such that

$$\sin^2\theta = \frac{4bc}{(b+c)^2}\cos^2\frac{A}{2},$$

thus $$a^2 = (b+c)^2(1-\sin^2\theta) = (b+c)^2\cos^2\theta,$$

therefore $$a = (b+c)\cos\theta,$$

therefore $\log a = \log(b+c) + \log\cos\theta = \log(b+c) + L\cos\theta - 10;$

thus a is determined.

It is usual to give the name of *subsidiary angle* to an angle introduced into an expression for the purpose of putting it in the form of a product of factors. Thus θ in the preceding investigation is a *subsidiary angle.* We are certain that an angle exists which has the square of its sine equal to the given expression; for that expression is positive, and it is less than unity because $4bc$ is never greater than $(b+c)^2$ and $\cos^2\frac{A}{2}$ is less than unity. The equation for determining θ gives by taking logarithms

$$2\log\sin\theta = \log 4 + \log b + \log c - 2\log(b+c) + 2\log\cos\frac{A}{2},$$

therefore $2L\sin\theta = 2\log 2 + \log b + \log c - 2\log(b+c) + 2L\cos\frac{A}{2}$.

232. The process of Art. 229 is sometimes facilitated by the use of a *subsidiary* angle when the logarithms of b and c are known.

We have $\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2}$.

Now let $\frac{b}{c} = \tan\theta$; therefore

$$\frac{b-c}{b+c} = \frac{\tan\theta - 1}{\tan\theta + 1} = \tan\left(\theta - \frac{\pi}{4}\right);$$

thus $$\tan\frac{1}{2}(B-C) = \tan\left(\theta - \frac{\pi}{4}\right)\cot\frac{A}{2}.$$

Or thus, suppose c less than b; let $c = b\cos\phi$;

therefore $$\frac{b-c}{b+c} = \frac{1-\cos\phi}{1+\cos\phi} = \tan^2\frac{\phi}{2};$$

thus $$\tan\frac{1}{2}(B-C) = \tan^2\frac{\phi}{2}\cot\frac{A}{2}.$$

233. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let a and b be the given sides, and A the given angle;

$$\text{then } \frac{\sin B}{\sin A} = \frac{b}{a}; \text{ therefore } \sin B = \frac{b}{a}\sin A;$$

now if $\frac{b\sin A}{a}$ is less than unity, two different angles may be found less than 180° which have $\frac{b \sin A}{a}$ for sine, one of these angles being less than a right angle, and the other greater. If a be greater than b, then A must be greater than B, and therefore B must be an *acute* angle; thus only the smaller value is admissible for B. If a be less than b, then either value may be taken for B. When B is determined, C is known since it is $180° - A - B$, and then c can be found from

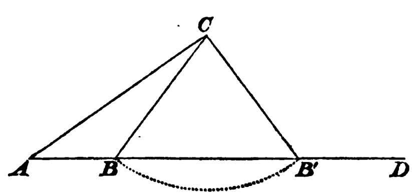
$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

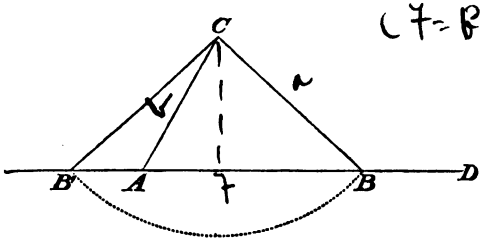
Thus if two values are admissible for B we obtain two corresponding values for C and c, so that *two* triangles can be found from the given parts.

If $\frac{b \sin A}{a} = 1$, then B is a right angle, so that only one triangle can be found from the given parts; and if $\frac{b \sin A}{a}$ is greater than unity, no triangle exists with the given parts.

Thus, when two sides are given and the angle opposite the less we can *generally* find two triangles from the given parts, and this case in the solution of triangles is therefore called the *ambiguous case.* We say that two triangles can be *generally* found in order to have regard to the exceptions; for the triangle may be *right angled*, and then only one triangle can be found, or the triangle may be *impossible.*

234. The *ambiguous case* may be illustrated by figures.





Let CAD be the given angle A, and AC the given side b; suppose a circle described from C as a centre with radius equal to a. The perpendicular from C on AD is equal to $b \sin A$; therefore if a be greater than $b \sin A$, the circle will meet the line AD in two points, which we will denote by B and B'. If a be less than b, then B and B' are on the same side of A, as in the first figure; thus two triangles, namely ABC and $AB'C$, can be obtained, each having the given parts a, b, A. If a be greater than b, then B' and B are on opposite sides of A, as in the second figure; thus only one triangle, namely CAB, can be obtained having the given parts a, b, A; the triangle CAB' has an angle CAB' which is $180^\circ - A$ instead of A.

If a be *equal* to $b \sin A$, the circle *touches* the line AD, and the two points B and B' in the first figure coincide; thus one triangle is obtained which has a right angle at B.

If a be *less* than $b \sin A$ the circle does not meet the line AD, and no triangle exists with the given parts a, b, A.

235. In Art. 233 we first found the angle B, and afterwards the side c; we may however adopt another mode of solution and begin by finding c. For

$$a^2 = b^2 + c^2 - 2bc \cos A;$$

therefore
$$c^2 - 2bc \cos A + b^2 - a^2 = 0;$$

by solving this quadratic equation in c we obtain

$$c = b \cos A \pm \sqrt{(a^2 - b^2 \sin^2 A)},$$

and we shall now discuss the values thus found for c.

If a is *less* than $b \sin A$, the values of c are impossible, and no triangle exists with the given parts.

If a is *equal* to $b \sin A$, we obtain $c = b \cos A$. If A be an acute angle, c is positive and one triangle exists with the given parts. If A be an obtuse angle, c is *negative*, and this indicates that the triangle is impossible; and in fact a is less than b, since it is equal to $b \sin A$, and so A cannot be an obtuse angle in a real triangle.

If a is *greater* than $b \sin A$, then two values occur for c, and these will both be positive if A be an acute angle and $b \cos A$ greater than $\sqrt{(a^2 - b^2 \sin^2 A)}$; the latter leads to the condition $b^2 \cos^2 A$ greater than $a^2 - b^2 \sin^2 A$, that is, b^2 greater than a^2. Hence we see as before that there are two triangles if A be an acute angle, and a be greater than $b \sin A$ and less than b.

236. *To solve a triangle having given the three sides.*

Let s denote half the sum of the sides; then by Art. 217,

$$\sin \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}, \quad \cos \frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}},$$

$$\tan \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}};$$

and similar formulæ are true for the other half angles.

The formulæ for the *tangents* of half the angles will be the best to use with logarithms, because then we only require the logarithms of s, $s-a$, $s-b$, and $s-c$, in order to find *all* the angles; whereas if we use the formulæ for the sine or cosine we shall require in addition the logarithms of the sides.

237. When all the sides of a triangle are given, the angles may also be found by dividing the triangle into two right-angled triangles.

Thus, with the left-hand figure of Art. 214, we have

$$AD^2 = AB^2 - BD^2, \text{ and also } = AC^2 - CD^2;$$

therefore $$AB^2 - AC^2 = BD^2 - CD^2,$$

therefore $(AB + AC)(AB - AC) = (BD + CD)(BD - CD)$;

from this we can find $BD-CD$, and then since $BD+CD$ is known we can find BD and CD; then

$$\cos B=\frac{BD}{AB},\quad \cos C=\frac{CD}{AC};$$

thus B and C are determined.

With the right-hand figure of Art. 214 we have as before

$$(AB+AC)(AB-AC)=(BD+CD)(BD-CD);$$

from this we can find $BD+CD$, and then since $BD-CD$ is known we can find BD and CD; then

$$\cos B=\frac{BD}{AB},\quad \cos(180^\circ-C)=\frac{CD}{AC};$$

thus B and C are determined.

238. We have seen in Chap. XII. that the Tables of trigonometrical functions cannot always be used with advantage; this circumstance guides us in selecting the method of solution of a triangle to be adopted when more than one method is theoretically applicable, and leads us to modify the method of solution in some cases. For example, suppose we have to find A from the equation $\sin A=n$, where n is nearly equal to unity; this is an inconvenient equation for determining A, because the difference of consecutive sines is nearly insensible when the angles are nearly right angles. We have however

$$\sin\left(45^\circ-\frac{A}{2}\right)=\sqrt{\left\{\frac{1-\cos(90^\circ-A)}{2}\right\}}$$

$$=\sqrt{\left(\frac{1-\sin A}{2}\right)}=\sqrt{\left(\frac{1-n}{2}\right)};$$

and this formula is free from the objection.

Similarly, if we have to find A from the equation

$$\cos A=n,$$

where n is nearly equal to unity, we may advantageously transform the equation thus,

$$\sin\frac{A}{2} = \sqrt{\left(\frac{1-\cos A}{2}\right)} = \sqrt{\left(\frac{1-n}{2}\right)};$$

or thus, $$\frac{1-\cos A}{1+\cos A} = \frac{1-n}{1+n};$$

therefore $$\tan\frac{A}{2} = \sqrt{\left(\frac{1-n}{1+n}\right)}.$$

EXAMPLES.

1. Find the values of the angle A having given $\sin B = \cdot 25$, $a = 5$, $b = 2\cdot 5$.

2. One side of a triangle is half another and the included angle is 60°; find the other angles.

3. The sides of a triangle are in the ratio of $2 : \sqrt{6} : 1 + \sqrt{3}$; determine the angles.

4. If $A = 30^\circ$, $b = 100$, $a = 40$, is there any ambiguity?

5. Having given $A = 18^\circ$, $a = 4$, $b = 4 + \sqrt{(80)}$, solve the triangle.

6. Having given $A = 15^\circ$, $a = 4$, $b = 4 + \sqrt{(48)}$, solve the triangle.

7. If a, b, A be given, and a be less than b, and if c, c' be the two values found for the third side of the triangle, then

$$c^2 - 2cc' \cos 2A + c'^2 = 4a^2 \cos^2 A.$$

8. Find the sum of the areas of the two triangles which satisfy the conditions of the problem in the *ambiguous case.*

9. If B_1, C_1, and B_2, C_2 are the angles of the two triangles in the *ambiguous case,* then

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2 \cos A.$$

10. In the *ambiguous case* the area of one of the triangles is n times that of the other; shew that if b be the greater of the given sides and a the less, $\frac{b}{a}$ is greater than 1 and less than $\frac{n+1}{n-1}$.

11. If $\log a + 10 = \log b + L \sin A$, can the triangle be ambiguous?

12. If θ be an angle determined from the equation

$$\cos\theta = \frac{a-b}{c},$$

prove that in any triangle

$$\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{(ab)}}, \quad \cos\frac{A+B}{2} = \frac{c\sin\theta}{2\sqrt{(ab)}}.$$

13. If $\tan\phi = \frac{2\sqrt{(ab)}}{a-b}\sin\frac{C}{2}$, then $c = (a-b)\sec\phi$.

14. In a triangle ABC in which $a = 18$, $b = 20$, $c = 22$, find $L\tan\frac{A}{2}$, having given

$$\log 2 = \cdot 3010300, \ \log 3 = \cdot 4771213.$$

15. The sides of a triangle are 32, 40, 66; find the greatest angle, having given

$$\log 207 = 2\cdot 3159703, \quad \log 1073 = 3\cdot 0305997,$$
$$L\cot 66^\circ\, 18' = 9\cdot 6424342, \quad \text{diff. for } 1' = \cdot 0003433.$$

16. The sides of a triangle are 4, 5, 6; find B, having given $\log 2 = \cdot 3010300$,

$$L\cos 27^\circ\, 53' = 9\cdot 9464040, \quad \text{diff. for } 1' = \cdot 0000669.$$

17. Apply the formula $\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$ to find the greatest angle in a triangle whose sides are 5, 6, 7 feet respectively, having given

$$\log 6 = \cdot 7781513,$$
$$L\cos 39^\circ\, 14' = 9\cdot 8890644, \quad \text{diff. for } 60'' = \cdot 0001032.$$

18. Two sides of a triangle are 18 and 2 feet respectively, and the included angle is $55°$; find the remaining angles, having given

$$\log 2 = \cdot 3010300, \quad L \cot 27° 30' = 10\cdot 2835233,$$
$$L \tan 56° 56' = 10\cdot 1863769, \quad \text{diff. for } 1' = \cdot 0002763.$$

19. Two sides of a triangle are in the ratio of 9 to 7, and the included angle is $64° 12'$; find the other angles, having given

$$\log 2 = \cdot 3010300, \quad L \tan 57° 54' = 10\cdot 2025255,$$
$$L \tan 11° 16' = 9\cdot 2993216, \quad L \tan 11° 17' = 9\cdot 2999804.$$

20. If $a = 70$, $b = 35$, $C = 36° 52' 12''$, find the remaining angles, having given

$$\log 3 = \cdot 4771213, \quad L \cot 18° 26' 6'' = 10\cdot 4771213.$$

21. The ratio of two sides of a triangle is 9 to 7, and the included angle is $47° 25'$; find the other angles, having given

$$\log 2 = \cdot 3010300, \quad L \tan 66° 17' 30'' = 10\cdot 3573942,$$
$$L \tan 15° 53' = 9\cdot 4541479, \quad \text{diff. for } 1' = \cdot 0004797.$$

22. In a triangle ABC where $a = 30$, $b = 20$, and the contained angle $= 22°$; find the other angles, having given

$$L \cot 11° = 10\cdot 7113477, \quad L \tan 45° 48' = 10\cdot 0121294,$$
$$L \tan 45° 49' = 10\cdot 0123821, \quad \log 2 = \cdot 3010300.$$

23. Given $b = 14$, $c = 11$, $A = 60°$, shew that $B = 71° 44' 29''$, having given $L \tan 11° 44' 29'' = 9\cdot 31774$,

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 47712.$$

24. The sides of a triangle are 7, 8, 9; determine all the angles, having given

$$\log 2 = \cdot 3010300,$$
$$L \tan 24° 5' 40'' = 9\cdot 6505069, \quad L \tan 24° 5' 50'' = 9\cdot 6505634,$$
$$L \tan 29° 12' 20'' = 9\cdot 7474183, \quad L \tan 29° 12' 30'' = 9\cdot 7474677.$$

25. In a right-angled triangle the hypothenuse $c = 6953$ and $b = 3$; find B, having given

$$\log 3{\cdot}475 = {\cdot}5409548, \quad \log 6{\cdot}953 = {\cdot}8421722,$$

$$L \sin 44^\circ\, 59'\, 15'' = 9{\cdot}8493902, \text{ diff. for } 1'' = {\cdot}0000021.$$

26. Two sides are 80 and 100 feet, and the included angle 60°; find the other angles, having given

$$\log 3 = {\cdot}47712, \quad L \tan 10^\circ\, 53'\, 36'' = 9{\cdot}28432.$$

27. Two sides of a triangle are 3 and 5 feet, and the included angle is 120°; find the other angles, having given

$$\log 4{\cdot}8 = {\cdot}6812412,$$

$$L \tan 8^\circ\, 12' = 9{\cdot}1586706, \text{ diff. for } 60'' = {\cdot}0008940.$$

28. A side of a base of a square pyramid is 200 feet and each edge is 150 feet; find the slope of each face, having given

$$\log 2 = {\cdot}30103, \quad L \tan 26^\circ\, 33' = 9{\cdot}69868,$$

$$L \tan 26^\circ\, 34' = 9{\cdot}69900.$$

29. Given $\frac{a}{b} = 1{\cdot}\dot{2}$, $C = 60^\circ$, $\log 3 = {\cdot}4771213$, $L \cot 9^\circ\, 49' = 10{\cdot}7618797$, diff. for $1' = {\cdot}0007514$, find the other angles.

30. If $a = 2$, $c = 3$, $L \sin A = 9{\cdot}5228787$, find C; $\log 3$ being ${\cdot}4771213$.

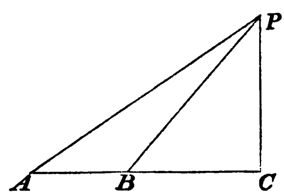
31. Shew how to solve a triangle having given the base, the height, and the difference of the angles at the base.

32. Shew how to solve a triangle having given the three perpendiculars from the angles on the opposite sides.

XV. ON THE MEASUREMENT OF HEIGHTS AND DISTANCES.

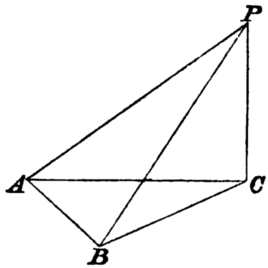
239. We shall now give a few examples which will shew a practical application of some of the preceding formulæ; we shall assume that by means of suitable instruments an observer can measure the angle subtended at his eye by the line joining two visible objects. For a description of the requisite instruments, and the method of using them, we must refer the student to treatises on the instruments used in surveying.

240. *To find the height and distance of an inaccessible object on a horizontal plane.*



Let P be the top of an object, and let it be required to find its height PC, and the distance of the object from a point A in the horizontal plane through C. At A observe the angle PAC; then measure any length AB directly towards the object, and at B observe the angle PBC. Then in the triangle APB the side AB is known, and the angle PAB; also the angle PBA is known, since it is the supplement of PBC; therefore AP can be found. Then $PC = AP \sin PAC$, and $AC = AP \cos PAC$; thus the height PC and the distance AC are determined.

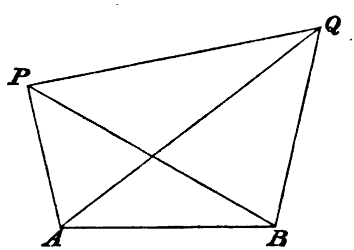
If however it is not convenient to measure the length AB directly towards the object, we may proceed thus; measure the length AB in *any* direction from A; at A observe the angles PAC



and PAB, and at B observe the angle PBA. Then in the triangle APB the side AB and the angles PAB and PBA are known; therefore AP can be found. Then, as before, $PC = AP \sin PAC$, and $AC = AP \cos PAC$.

241. *To find the distance between two visible but inaccessible objects.*

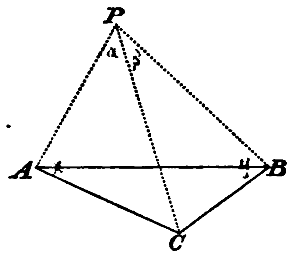
Let P and Q be the objects, A and B two accessible points from which both the objects are visible. At A observe the angles PAQ and QAB, and if A, B, Q, P are not all in the same plane observe also the angle PAB. At B observe the angles PBA and QBA. Measure AB. Then in the triangle ABP the side AB and the angles PAB and PBA are known; thus PA can be found. Again, in the triangle ABQ the side AB and the angles QAB and



QBA are known; thus AQ can be found. Lastly, in the triangle PAQ the sides AP, AQ, and the angle PAQ are known; thus PQ can be found.

242. *The lengths of the lines which join three points* A, B, C *are known; at any point* P *in the same plane as* A, B, C, *the angles* APC *and* BPC *are observed: it is required to find the distance of* P *from each of the points* A, B, C.

Let the angle APC be denoted by α the angle BPC by β; the angle PAC by x, and the angle PBC by y; then α and β are



known, and when x and y are found the required distances PA, PB, PC can be found; for in each of the triangles PAC and PBC two angles and a side will then be known. We will shew how x and y may be found.

Since the four angles of the quadrilateral $PACB$ are together equal to four right angles, we have

$$x + y = 2\pi - \alpha - \beta - C;$$

thus the *sum* of x and y is known.

From the triangle ACP we have

$$PC = \frac{AC \sin PAC}{\sin APC} = \frac{b \sin x}{\sin \alpha};$$

from the triangle BCP we have

$$PC = \frac{BC \sin PBC}{\sin BPC} = \frac{a \sin y}{\sin \beta};$$

therefore $$\frac{b \sin x}{\sin \alpha} = \frac{a \sin y}{\sin \beta};$$

therefore $$\frac{\sin x}{\sin y} = \frac{a \sin \alpha}{b \sin \beta}.$$

Now assume $\tan \phi = \dfrac{a \sin \alpha}{b \sin \beta}$, then the value of ϕ can be found from the Trigonometrical Tables; thus

$$\frac{\sin x}{\sin y} = \tan \phi;$$

therefore $$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \phi - 1}{\tan \phi + 1} = \tan\left(\phi - \frac{\pi}{4}\right);$$

therefore (Art. 88) $$\frac{\tan \frac{1}{2}(x-y)}{\tan \frac{1}{2}(x+y)} = \tan\left(\phi - \frac{\pi}{4}\right);$$

from the last equation we can determine $x-y$, since $x+y$ is known; thus x and y can be found.

243. It is sometimes important to know what amount of error will be introduced into one of the calculated parts of a triangle by reason of any error which may exist in the given parts; such questions are best treated by the assistance of the Differential Calculus, but we will give here two simple examples which will shew how they may sometimes be treated without going beyond the limits of the present subject.

244. Suppose that the height of a building is determined by measuring a horizontal line from its base, and by observing at the extremity of this line the angular elevation of the top of the building above the horizon; if a small error be made in observing the angle, required the error in the estimated height of the building.

Let a be the length of the measured line, θ the observed angle, x the estimated height of the building;

then $$x = a \tan \theta.$$

Let $\theta + h$ be the true angle, and $x + \xi$ the true height,

then $$x + \xi = a \tan (\theta + h);$$

by subtraction, $\xi = a \{\tan (\theta + h) - \tan \theta\} = \dfrac{a \sin h}{\cos (\theta + h) \cos \theta}$.

If h be small we may put h for $\sin h$ in the numerator, and $\cos \theta$ for $\cos (\theta + h)$ in the denominator; thus approximately

$$\xi = \frac{ah}{\cos^2 \theta};$$

this gives the error in the height consequent upon an error in the angle.

The *ratio* of the error to the estimated height

$$= \frac{ah}{\cos^2 \theta} \div a \tan \theta = \frac{h}{\sin \theta \cos \theta} = \frac{2h}{\sin 2\theta};$$

thus this ratio is least for a given value of h when $\sin 2\theta$ is greatest, that is, when $2\theta = \dfrac{\pi}{2}$.

245. A triangle is solved from the given parts A, b, c; if there be a small error in A, find the consequent small error in B.

We have for connecting B with the given quantities the formula

$$\sin B = \frac{b}{c} \sin C = \frac{b}{c} \sin (A + B) \qquad \text{.............(1).}$$

Now suppose that h denotes the *circular measure* of the error made in estimating A, and k the *circular measure* of the consequent error in B; then instead of (1), the correct formula is

$$\sin (B + k) = \frac{b}{c} \sin (A + B + h + k) \qquad \text{.........(2).}$$

By subtraction,

$$\sin(B + k) - \sin B = \frac{b}{c} \{\sin (A + B + h + k) - \sin (A + B)\};$$

from this equation we have approximately (Art. 181)

$$k \cos B = \frac{b}{c}(h+k)\cos(A+B) = -\frac{b}{c}(h+k)\cos C;$$

thus $$k\left(\cos B + \frac{b}{c}\cos C\right) = -\frac{bh}{c}\cos C;$$

therefore $$k\left(\cos B + \frac{\sin B}{\sin C}\cos C\right) = -\frac{h \sin B \cos C}{\sin C};$$

therefore $$k = -\frac{h \sin B \cos C}{\sin A},$$

thus the ratio of k to h is found.

EXAMPLES.

1. From a station B at the base of a mountain its summit A is seen at an elevation of $60°$; after walking one mile towards the summit up a plane making an angle of $30°$ with the horizon to another station C, the angle BCA is observed to be $135°$. Find the height of the mountain in yards.

2. The altitude of a tower is observed to be $30°$ at the end of a horizontal base of 100 yards measured from its foot. Find the height of the tower.

3. The angular elevation of a tower at a place A due south of it is $30°$; and at a place B, due west of A, and at the distance a from it, the elevation is $18°$; shew that the height of the tower is

$$\frac{a}{\sqrt{(2+2\sqrt{5})}}.$$

4. A person on a level plain, on which stands a tower surmounted by a spire, observes that when he is a feet distant from the foot of the tower its top is in a line with that of a mountain. From a point b feet farther from the tower he finds that the spire subtends at his eye the same angle as before, and has its top in a

line with that of the mountain. Shew that if the height of the tower above the horizontal plane through the observer's eye be c feet, the height of the mountain above that plane will be $\frac{abc}{c^2 - a^2}$ feet.

5. A person wishing to ascertain his distance from an inaccessible object finds three points in the horizontal plane at which the angular elevation of the summit of the object is the same. Shew how the distance may be found.

6. A person wishing to ascertain the distances between three inaccessible objects A, B, C, places himself in a line with A and B; he then measures the distances along which he must walk in a direction at right angles to AB until A, C and B, C respectively are in a line with him, and also observes in those positions their angular bearings; shew how he can find the distances between A, B, C.

7. Two posts AB and CD are placed at the edge of a river at a distance $AC = AB$, the height of CD being such that AB and CD subtend equal angles at E, a point on the other bank exactly opposite to A; shew that the square of the breadth of the river is equal to $\frac{AB^4}{CD^2 - AB^2}$, and that AD and BC subtend equal angles at E.

8. A flag-staff a feet high stands on the top of a tower b feet high. At what point on a horizontal plane passing through the base of the tower must an observer place himself so that the tower and the flag-staff may subtend equal angles, the height of the eye being h?

9. A tower situated on a horizontal plane leans towards the north; at two points due south and distant a, b, respectively from the base, the angular altitudes of the tower are α and β. Shew that if θ be the inclination of the tower, and h the perpendicular height,

$$\tan\theta = \frac{b - a}{b\cot\alpha - a\cot\beta}, \quad h = \frac{b - a}{\cot\beta - \cot\alpha}.$$

10. An object a feet high placed on the top of a tower subtends an angle γ at a place whose horizontal distance from the foot of the tower is b feet; determine the height of the tower.

11. On the bank of a river there is a column 200 feet high supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column; required the breadth of the river.

12. The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal line; find the height, taking the breadth of the street 30 feet.

13. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to him to be 60°. After walking 80 feet in a direction at right angles to the line joining their bases he observes the elevations of the two to be respectively 45° and 30°. Find their height and the distance between them.

14. An object is observed at three points A, B, C lying in a horizontal line which passes directly underneath the object; the angular elevation at B is twice that at A, and at C is three times that at A; $AB = a$, $BC = b$; shew that the height of the object is

$$\frac{a}{2b}\sqrt{\{(a+b)(3b-a)\}}.$$

If the tangent of the angle of elevation at A be $\frac{1}{3}$, shew that $5a = 13b$.

15. A vertical tower whose base is in the same horizontal plane with the observer, is observed from a station A to bear directly North and to subtend an angle of 15°; the observer then walks 100 yards so that the tower always subtends the same angle, and then it bears North-east; find its height and distance from A.

16. A person walking along a straight road observes that the greatest angle which two objects subtend is α; from the spot where this is the case he walks a distance c, and the objects now

appear as one, their direction making an angle β with the road. Prove that the distance between the objects is

$$\frac{2c \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}.$$

17. A fortress was observed by a ship at sea to bear E.N.E., and after sailing 4 miles to the East it was observed to bear N.N.E.; shew that the distance of the ship from the fortress at the first and second observation was $\sqrt{(16 + 8\sqrt{2})}$ and $\sqrt{(16 - 8\sqrt{2})}$ miles respectively.

18. A ship sailing towards the North observes two lighthouses in a line due West; and after an hour's sailing the bearings of the lighthouses are observed to be South-west and South-south-west. The distance between the lighthouses being 8 miles, find the rate at which the ship is sailing.

19. From the top of the mast of a ship 64 feet above the level of the sea the light of a distant lighthouse is just seen in the horizon; and after the ship has sailed directly towards the light for 30 minutes it is seen from the deck of the ship, which is 16 feet above the sea. Find the rate at which the ship is sailing, considering the earth as a sphere of 4000 miles radius.

20. A man ascends a mountain by a path which is the shortest distance between the base and the vertex. The inclination of the path to the horizon at first is α, but afterwards suddenly increases to β, and then continues the same. On reaching the vertex he finds by the barometer he has ascended n feet in altitude, and observes the angle of depression γ of the point from which he started. Shew that the distance he travelled in the ascent is

$$\frac{n \cos\left(\dfrac{\alpha + \beta}{2} - \gamma\right)}{\cos \dfrac{\beta - \alpha}{2} \sin \gamma}.$$

21. If from two points in a horizontal plane an object be seen at angles of elevation α, α', and if from a third point between

the two points and in the straight line joining them and at distances a, a' from them respectively the object be seen at an angle of elevation β, shew that the height of the object above the horizontal plane is

$$\frac{\sin\alpha\sin\alpha'\sin\beta\{aa'(a+a')\}^{\frac{1}{2}}}{\{a\sin^2\alpha(\sin^2\beta-\sin^2\alpha')+a'\sin^2\alpha'(\sin^2\beta-\sin^2\alpha)\}^{\frac{1}{2}}}.$$

22. A person walking along a straight road observes the angles of elevation α, α' of the summits of two hills in front of him, one behind and partially hid by the other. After walking c miles the farther hill becomes entirely hidden, and on observing the elevation of the lower hill at the next mile-stone he finds it to be β. Find the heights of the two hills.

23. A tower is surrounded by a circular moat. At noon on a certain day the shadow of the top of the tower is observed to project 45 feet beyond the edge of the moat. When the sun is due West on the same day the shadow projects 120 feet beyond the moat. The distance between the extremities of the shadow is 375 feet. The angle of elevation of the top of the tower from any point of the edge of the moat is 60°. Find the height of the tower and the altitude of the sun at noon.

24. A tower stands upon an inclined plane, meeting it at a point A; at a point C in the plane the tower is observed to subtend an angle α; on proceeding to a point D in the line AC such that $CD=AC$, the tower is observed to subtend an angle β; if ϕ be the angle between the tower and AC, shew that $\cot\phi=2\cot\alpha-\cot\beta$.

Also if similar observations be made in another line $AC'D'$, it is found that $\tan\alpha'=2\tan\beta'$; the angle $CAC'=\gamma$; prove that if θ be the inclination of the plane to the horizon, $\sin\theta\sin\gamma=\cos\phi$.

25. In a triangle ABC having given $A=30°$, $b=3\sqrt{3}$, $a=3$, solve the triangle; and supposing that an error of $2''$ is made in observing the angle A, find approximately the corresponding error in the angle B.

26. The distance between two objects on the opposite bank of a river is known to be c. An equal distance is taken anywhere along the bank on this side and the angles subtended by c at the extremities of this distance are α and β. Find the breadth of the river, the sides being parallel.

27. A person wishing to obtain the breadth of a square fort on a distant hill, observes that when he is due South of one corner, the face towards him subtends an angle α. He then walks due West, and at a distance of a feet from his first position, finds that the face subtends the same angle as before. On walking b feet further, he is due South of the other corner of the face. Shew that the breadth of the fort is

$$(a+b)\sec\phi \text{ feet, where } \tan\phi = \frac{b\tan\alpha}{a+b}.$$

28. A and A' are the peaks of two mountains, and BC is a straight horizontal road; shew that if the nearer of the two peaks just conceals the more distant at some point of the road, then $\sin\alpha\sin\beta' = \sin\alpha'\sin\beta$, where α is the altitude of A as seen from any point B of the road, β is the angle ABC, and α', β' are similar quantities for the peak A' as seen from any point B' of the road.

29. A and B are two objects in the same horizontal plane, P a point in the same plane at which the angle α subtended by AB is observed; from P two persons walk in this plane in directions at right angles to PA, PB respectively, to points Q, R, at each of which the angle subtended by AB is α; the distances PQ, PR are a, b; find the length of AB.

30. A, C, B are three objects in the same plane as an observer; $AC = CB$, and AC, CB are at right angles to each other. At the point O, AC, CB subtend angles α, β respectively. The observer moves from O in the direction OO' at right angles to CO through a space $OO' = d$; here he finds that AC, CB subtend angles α', β' respectively. Find the distance AB.

31. A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an

angle of 55° with a horizontal line drawn from his eye; receding backwards 30 feet he then finds it to subtend an angle of 48°. Determine the breadth of the river, having given

$$L\sin 7^\circ = 9\cdot08589, \quad L\sin 35^\circ = 9\cdot75859,$$
$$L\sin 48^\circ = 9\cdot87107, \quad \log 3 = \cdot47712,$$
$$\log 1\cdot0493 = \cdot02089.$$

32. A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane upon which it stands. Find the Sun's altitude, having given

$$\log 2 = \cdot3010300, \quad L\tan 63^\circ 26' = 10\cdot3009994,$$
$$L\tan 63^\circ 27' = 10\cdot3013153.$$

33. A rope-dancer wishes to ascend a tower 100 feet high, by means of a rope 196 feet long. If he can do so, find at what inclination he must be able to walk up the rope, having given

$$\log 2 = \cdot30103, \quad L\sin 30^\circ 40' = 9\cdot70761,$$
$$\log 7 = \cdot84510, \quad L\sin 30^\circ 41' = 9\cdot70782.$$

34. Two hills rise at the same point, with inclinations of 60° and 40° to the horizon. At a distance of 64 feet from the base of the lower hill the angles of elevation of the bottom and top of a vertical object on the other hill are 40° and 70°. Find the height of the object, having given

$$L\tan 20^\circ = 9\cdot5610659, \quad L\cos 40^\circ = 9\cdot8842540,$$
$$\log 2 = \cdot3010300; \quad 7\cdot4303981 = \log 29640031.$$

35. A vessel observed another α° from the North sailing in a direction parallel to its own. After an hour's sailing its bearing was β°, and after another hour γ° from the North. In what direction were the vessels sailing?

36. In the problem discussed in Art. 242, shew that if

$$\alpha + \beta + C = \pi, \text{ then } \phi = \frac{\pi}{4},$$

and the solution cannot be obtained from the data.

XVI. PROPERTIES OF TRIANGLES.

246. The present chapter will contain some miscellaneous propositions relating chiefly to the properties of triangles.

247. *To find expressions for the area of a triangle.*

A triangle is half a rectangle on the same base and altitude; thus if ABC be any triangle, and AD the perpendicular from A on the opposite side, we have (see the figures in Art. 214)

$$\text{area of triangle} = \tfrac{1}{2} BC \,.\, AD,$$

and $$AD = AB \sin B,$$

therefore $$\text{area of triangle} = \tfrac{1}{2} ac \sin B \ldots\ldots\ldots\ldots\ldots(1);$$

thus the area of a triangle is half the product of two sides into the sine of the included angle.

By Art. 218, $\sin B = \dfrac{2}{ac} \sqrt{\{s(s-a)(s-b)(s-c)\}}$;

substitute the value of $\sin B$ in (1) and we obtain

$$\text{area of triangle} = \sqrt{\{s(s-a)(s-b)(s-c)\}} \ldots\ldots\ldots\ldots\ldots(2);$$

this furnishes a convenient expression for the area when all the sides are known; the expression $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ is often for abbreviation denoted by S.

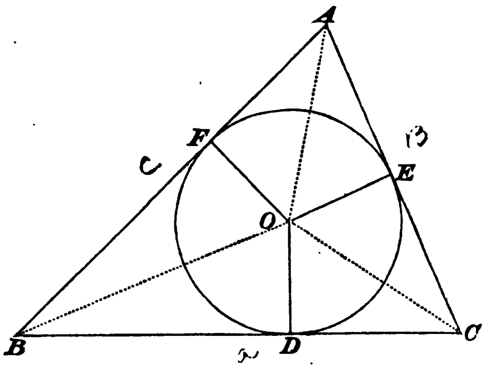
By Art. 214, $a = \dfrac{b \sin A}{\sin B}$, $\quad c = \dfrac{b \sin C}{\sin B}$,

substitute these values in (1); thus we obtain

$$\text{area of triangle} = \frac{b^2 \sin A \sin C}{2 \sin B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

thus we can find the area when a side and two angles are given, for if two angles are given the third angle is also known.

248. *To find the radius of the circle inscribed in a triangle.*



Let ABC be a triangle, O the centre of the circle inscribed in the triangle and touching the sides in the points D, E, F. Let r denote the radius of the circle; then

$$\text{area of triangle } BOC = \tfrac{1}{2} BC \,.\, OD = \frac{ar}{2},$$

$$\text{area of triangle } COA = \tfrac{1}{2} CA \,.\, OE = \frac{br}{2},$$

$$\text{area of triangle } AOB = \tfrac{1}{2} AB \,.\, OF = \frac{cr}{2};$$

therefore, by addition,

$$(a + b + c)\frac{r}{2} = \text{area of triangle } ABC = S, \quad \text{(Art. 247)},$$

therefore $$r = \frac{S}{s}.$$

The radius of the inscribed circle is thus equal to the area of the triangle divided by half the sum of the sides; and thus different forms can be obtained for the radius by employing the different expressions already given for the area of the triangle.

249. We may also obtain the value of r in another form, which will be often useful.

By Euclid IV. 4, the lines OA, OB, OC bisect the angles A, B, C respectively. Thus

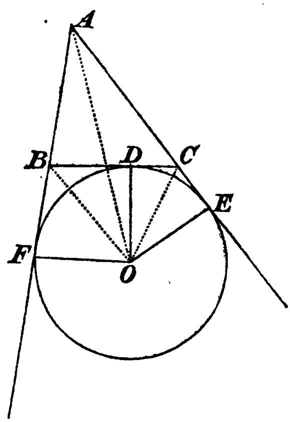
$$BD = r\cot\frac{B}{2}, \qquad CD = r\cot\frac{C}{2},$$

therefore $$r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = a,$$

therefore $$r\sin\frac{B+C}{2} = a\sin\frac{B}{2}\sin\frac{C}{2},$$

therefore $$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

250. *To find the radius of a circle which touches one side of a triangle and the other sides produced.*



Let ABC be a triangle, and let O be the centre of the circle which touches the side BC, and the other sides produced. Let r_1 denote the radius of the circle.

The quadrilateral $OBAC$ may be divided into the two triangles OAB, OAC; therefore the area of this quadrilateral is $\frac{c}{2}r_1 + \frac{b}{2}r_1$. Again, the same quadrilateral may be divided into the triangles OBC and ABC; therefore the area of this quadrilateral is $\frac{a}{2}r_1 + S$. Thus

$$\frac{c}{2}r_1 + \frac{b}{2}r_1 = \frac{a}{2}r_1 + S;$$

therefore $$\frac{r_1(c+b-a)}{2} = S,$$

therefore $$r_1 = \frac{S}{s-a}.$$

Similarly, if r_2 be the radius of the circle which touches CA and the other sides produced, and r_3 the radius of the circle which touches AB and the other sides produced,

$$r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

A circle which touches one side of a triangle and the other sides produced is called an *escribed* circle.

251. We may also obtain an expression for the radius of an escribed circle similar to that in Art. 249 for the radius of the inscribed circle.

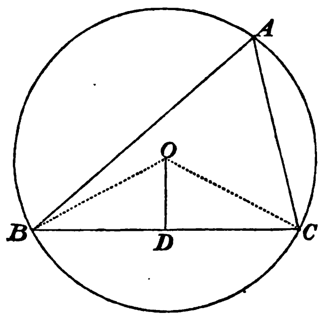
For, in the figure of Art. 250, the line OB bisects the angle which is the supplement of B, and the line OC bisects the angle which is the supplement of C; thus

$$BD = r_1 \cot\left(90^\circ - \frac{B}{2}\right), \quad CD = r_1 \cot\left(90^\circ - \frac{C}{2}\right);$$

therefore $$r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = a;$$

therefore $$r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B+C}{2}} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

252. *To find the radius of the circle described round a triangle.*



Let ABC be a triangle, and O the centre of the circle described round it. Draw OD perpendicular to BC, then BC is bisected in D by Euclid IV. 5. Let R denote the radius of the circle.

The angle BOC is double the angle BAC; therefore

$$BOD = A;$$

and
$$BD = R \sin A = \frac{a}{2};$$

therefore
$$R = \frac{a}{2 \sin A};$$

thus R is expressed in terms of a side and the opposite angle. By Art. 218, $\sin A = \frac{2S}{bc}$, therefore

$$R = \frac{abc}{4S}.$$

253. Many theorems have been demonstrated with respect to the circles which have been noticed in Arts. 248—252; as an example we will find an expression for the distance between the centres of the inscribed and circumscribed circles.

Let O denote the centre of the circumscribed circle, and O' the centre of the inscribed circle; and suppose O and O' joined with the angular point C of the triangle. Then

$$OO'^2 = OC^2 + O'C^2 - 2OC \cdot O'C \cos OCO';$$

now the angle $O'CB = \frac{1}{2}C$, and the angle $OCB = 90^\circ - A$; thus

$$\cos OCO' = \cos\left(90^\circ - A - \frac{C}{2}\right)$$

$$= \cos\left(\frac{A+B+C}{2} - A - \frac{C}{2}\right) = \cos\frac{B-A}{2};$$

also $$OC = R, \quad O'C = \frac{r}{\sin\frac{C}{2}};$$

therefore $$OO'^2 = R^2 + \frac{r^2}{\sin^2\frac{C}{2}} - \frac{2Rr}{\sin\frac{C}{2}}\cos\frac{B-A}{2}.$$

By Art. 249, $$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}};$$

by Art. 252, $$R = \frac{a}{2\sin A};$$

therefore $$\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Therefore

$$OO'^2 = R^2 - \frac{2Rr}{\sin\frac{C}{2}}\left\{\cos\frac{B-A}{2} - 2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\right\}$$

$$= R^2 - \frac{2Rr}{\sin\frac{C}{2}}\left\{\cos\frac{B}{2}\cos\frac{A}{2} - \sin\frac{B}{2}\sin\frac{A}{2}\right\}$$

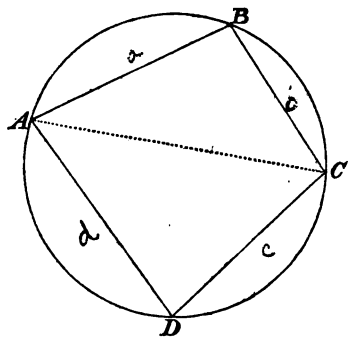
$$= R^2 - 2Rr.$$

Therefore $$OO' = \sqrt{(R^2 - 2Rr)}.$$

254. *To find the area of a quadrilateral which can be inscribed in a circle.*

Let $ABCD$ be the quadrilateral; let

$$AB = a, \quad BC = b, \quad CD = c, \quad DA = d.$$



The figure can be divided into the triangles ABC, ADC; its area therefore

$$= \tfrac{1}{2}(ab \sin B + cd \sin D) = \tfrac{1}{2}(ab + cd) \sin B,$$

for the angles B and D are supplemental.

Now from the triangle ABC,

$$AC^2 = a^2 + b^2 - 2ab \cos B,$$

and from the triangle CDA,

$$AC^2 = c^2 + d^2 - 2cd \cos D = c^2 + d^2 + 2cd \cos B;$$

therefore $\quad c^2 + d^2 + 2cd \cos B = a^2 + b^2 - 2ab \cos B,$

therefore $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)};$$

therefore $$\sin^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}$$

$$=\frac{\{2(ab+cd)+c^2+d^2-a^2-b^2\}\{2(ab+cd)-c^2-d^2+a^2+b^2\}}{4(ab+cd)^2}$$

$$=\frac{\{(c+d)^2-(a-b)^2\}\{(a+b)^2-(c-d)^2\}}{4(ab+cd)^2}$$

$$=\frac{(c+b+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}{4(ab+cd)^2}.$$

Now let $\frac{1}{2}(a+b+c+d)=s$; thus

$$\sin^2 B=\frac{16(s-a)(s-b)(s-c)(s-d)}{4(ab+cd)^2}.$$

Hence the area of the quadrilateral

$$=\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

If we substitute the value of $\cos B$ in the expression for AC^2, we obtain

$$AC^2=c^2+d^2+\frac{2cd(a^2+b^2-c^2-d^2)}{2(ab+cd)}$$

$$=c^2+d^2+\frac{cd(a^2+b^2-c^2-d^2)}{ab+cd}$$

$$=\frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Similarly it may be shewn that

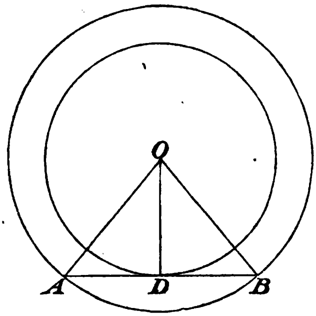
$$\cos A=\frac{a^2+d^2-b^2-c^2}{2(ad+bc)},$$

$$BD^2=\frac{(ac+bd)(ab+cd)}{ad+bc}.$$

The radius of the circle described round the quadrilateral may be easily expressed; for this circle passes round the triangle ABC, hence by Art. 252 its radius

$$=\frac{AC}{2\sin B}=\frac{1}{4}\sqrt{\left\{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}\right\}}.$$

255. *To find the radii of the inscribed and circumscribed circles of a regular polygon.*



Let AB be the side of a regular polygon of n sides; let O be the centre of the circles, OD the radius of the inscribed circle, OA the radius of circumscribed circle.

Let $AB = a$, $OA = R$, $OD = r$.

The angle AOB is the n^{th} part of 4 right angles, that is,

$$AOB = \frac{2\pi}{n}, \quad AOD = \frac{\pi}{n}.$$

$$AD = \frac{a}{2} = R \sin\frac{\pi}{n} = r \tan\frac{\pi}{n};$$

therefore $$R = \frac{a}{2\sin\frac{\pi}{n}}, \quad r = \frac{a}{2\tan\frac{\pi}{n}}.$$

256. The area of a regular polygon may be expressed by means of the radius of the inscribed circle, or the radius of the circumscribed circle. For with the figure of Art. 255, the area of the triangle AOB

$$= \tfrac{1}{2} AB \,.\, OD = \frac{a}{2}\,\frac{a}{2} \cot\frac{\pi}{n} = \frac{a^2}{4} \cot\frac{\pi}{n};$$

therefore the area of the polygon

$$=\frac{na^2}{4}\cot\frac{\pi}{n}$$

$$=nR^2\sin^2\frac{\pi}{n}\cot\frac{\pi}{n}=\frac{n}{2}R^2\sin\frac{2\pi}{n}.$$

Also the area of the polygon

$$=nr^2\tan^2\frac{\pi}{n}\cot\frac{\pi}{n}=nr^2\tan\frac{\pi}{n}.$$

257. *To find the area of a circle.*

The area of a regular polygon of n sides described about a circle of radius r

$$=nr^2\tan\frac{\pi}{n}=\frac{\pi r^2}{\cos\frac{\pi}{n}}\cdot\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}.$$

Now suppose n to increase without limit, then the area of the polygon approximates continually to the area of the circle as its limit, and therefore the area of the circle will be the limit of the above expression. But when n is indefinitely great,

$$\cos\frac{\pi}{n}=1,\quad \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=1,\ \text{(Art. 118)};$$

therefore *area of circle of radius* $r=\pi r^2$.

258. *To find the area of a sector of a circle.*

Let θ be the circular measure of the angle of the sector; then

$$\frac{\text{area of sector}}{\text{area of circle}}=\frac{\theta}{2\pi};$$

therefore $$\text{area of sector}=\pi r^2\times\frac{\theta}{2\pi}=\frac{r^2\theta}{2}.$$

Since θ is the circular measure of the angle of the sector, the length of the arc of the sector is $r\theta$; hence the area of a sector is equal to half the product of the length of the arc into the radius.

EXAMPLES.

1. The sides of a plane triangle are 24, 30, 18; find the area.

2. Two angles of a triangle are 15° and 45°, and the included side 10 feet; find the area.

3. The sides of a triangle are equal to 3 and 12 respectively, and the contained angle is 30°; find the hypotenuse of an equal right-angled isosceles triangle.

4. The area of a triangle $= \frac{1}{4}(a^2 \sin 2B + b^2 \sin 2A)$.

5. The area of a triangle $= \dfrac{a^2 - b^2}{2} \dfrac{\sin A \sin B}{\sin (A - B)}$.

6. The area of a triangle

$$= \frac{2abc}{a+b+c} \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

7. Shew that the triangle whose sides are proportional to

$$gh(k^2 + l^2), \quad kl(g^2 + h^2), \quad (hk + gl)(hl - gk)$$

has its area and the trigonometrical ratios of its angles rational.

8. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 to 5. Find the ratio of the sides and the value of the largest angle.

9. If the alternate angles of a regular hexagon be joined so as to form another regular hexagon, and again the alternate angles of the latter hexagon be joined, and so on, shew that the sum of the

areas of all the figures so formed $=\frac{A}{2}$, where A is the area of the original figure. And generally if the figure has n sides, the sum

$$=\frac{A\cos^2\frac{2\pi}{n}}{\sin\frac{3\pi}{n}\sin\frac{\pi}{n}}.$$

Explain the cases where $n=3$ or 4.

10. If an equilateral triangle be described with its angular points on the sides of a given right-angled isosceles triangle, and one side parallel to the hypotenuse, its area will be

$$2a^2\sin 60^\circ (\sin 15^\circ)^2,$$

where a is a side of the given triangle.

11. The distance between two points is a, and their distances from a given line are b, c; of all the triangles which can be formed having the same base a, and whose vertices lie on the given line, the area of that which has the greatest vertical angle is $\frac{a}{2}\sqrt{(bc)}$.

12. The straight lines which bisect the angles A, C of a triangle ABC meet the circumference of the circumscribing circle in the points A', C'; shew that $A'C'$ is divided by CB, BA into three parts, which are in the proportion

$$\sin^2\frac{A}{2} : 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \sin^2\frac{C}{2}.$$

13. If a be the difference between the sides containing the right angle of a right-angled triangle, and S its area, the diameter of the circumscribing circle is equal to $\sqrt{(a^2+4S)}$.

14. The sides of a plane triangle are 3, 5, 6; compare the radii of the inscribed and circumscribed circles.

15. O is the centre of the circle circumscribed round a triangle, and AO is produced to meet BC in D; shew that

$$DO\cos(B-C)=AO\cos A.$$

16. A circle is inscribed within a given triangle, and another triangle formed by joining the points of contact; within this latter triangle a circle is inscribed, and another triangle formed as before, and so on continually; shew that the triangles thus formed ultimately become equilateral.

17. The sum of the diameters of the inscribed and circumscribed circles of any plane triangle is equal to

$$a \cot A + b \cot B + c \cot C.$$

18. Perpendiculars are drawn from the angles A, B, C of a triangle on the opposite sides, and produced to meet the circumscribing circle; if those produced parts be α, β, γ respectively, prove that

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2 (\tan A + \tan B + \tan C).$$

19. A circle is inscribed in a triangle ABC, and smaller circles are described so as to touch this circle and the two sides of the triangle; find their radii.

20. In any triangle the area of the inscribed circle is to the area of the triangle as π to $\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

21. On each side of an acute-angled triangle as base an isosceles triangle is constructed, the sides of each being equal to the radius of the circumscribed circle; if the vertices of these be joined a triangle will be formed equal and similar to the original.

22. If R be the radius of the circumscribed circle of a triangle,

$$a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$$

23. O is the centre of the circle circumscribed about a triangle ABC; from O the perpendiculars OD, OE, OF are drawn to the sides; shew that

$$4 (OD^2 + OE^2 + OF^2) = a^2 \cot^2 A + b^2 \cot^2 B + c^2 \cot^2 C.$$

24. If r be the radius of the circle inscribed in a triangle, and r_a, r_b, r_c the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively; prove that

$$\sqrt{(r_a r_b)} + \sqrt{(r_b r_c)} + \sqrt{(r_c r_a)} = r.$$

25. Given the segments into which the base of a triangle is divided by the point of contact of the inscribed circle; find the greatest possible value of the radius of the inscribed circle.

26. If a triangle $A'B'C'$ be formed by joining the feet of the perpendiculars let fall from A, B, C upon the opposite sides, shew that $B'C' = R \sin 2A$, where R is the radius of the circle circumscribed about ABC.

27. Perpendiculars drawn from the angular points of a triangle to the opposite sides meet those sides in the points D, E, F; prove that if R and R_1 be the radii of the circles described about the triangles ABC and DEF respectively, and r_1 the radius of the circle inscribed in the latter triangle,

$$R_1 = \tfrac{1}{2} R, \text{ and } r_1 = 2R \cos A \cos B \cos C.$$

28. If r, r_1, r_2, r_3 denote the radii of the inscribed and escribed circles of a triangle, prove that

$$\tan^2 \frac{A}{2} = \frac{rr_1}{r_2 r_3}.$$

29. If A be the area of the circle inscribed in a triangle, A_1, A_2, A_3 the areas of the escribed circles, then

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}.$$

30. If the sides of a triangle be in arithmetical progression the perpendicular on the mean side from the opposite angle, and the radius of the circle which touches the mean side and the other two sides produced, are each equal to three times the radius of the inscribed circle.

31. The distances of the centre of the circle inscribed in a triangle from the centres of the three escribed circles are respectively proportional to

$$\sin\frac{A}{2}, \quad \sin\frac{B}{2}, \quad \text{and } \sin\frac{C}{2}.$$

32. Two similar triangles have a common escribed circle touching sides not homologous a_1, b_2; shew that

$$a_1 : a_2 = \sin B + \sin C - \sin A : \sin A + \sin C - \sin B.$$

33. If O_1, O_2, O_3 are the centres of the escribed circles of a triangle, then the area of the triangle $O_1O_2O_3$

$$= \text{area of triangle } ABC\left\{1 + \frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c}\right\}.$$

34. The centres of the three escribed circles of a triangle are joined; shew that the area of the triangle thus formed is $\frac{abc}{2r}$, where r is the radius of the inscribed circle of the original triangle.

35. A', B', C' are the centres of the escribed circles of a triangle; A', B', C' are joined so as to form a triangle; if r and r' be the radii of the circles inscribed in ABC and $A'B'C'$ respectively,

$$\frac{r'}{r} = \frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}.$$

36. If r be the radius of the circle inscribed in a triangle ABC, $2s$ the sum of the sides, r', $2s'$ similar quantities for the triangle which is formed by joining the centres of the escribed circles; shew that

$$\frac{rs}{r's'} = 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

37. Let a, a_1 be the distances of the angle A of a triangle from the centres of the inscribed circle, and the circle touching the side

a and the other two produced; β, β_1 similar quantities for the angle B; γ, γ_1 similar quantities for the angle C; shew that

$$\alpha\beta\gamma\,\alpha_1\beta_1\gamma_1 = (abc)^2,$$

$$\frac{bc}{\alpha_1^2} + \frac{ca}{\beta_1^2} + \frac{ab}{\gamma_1^2} = 1,$$

$$\alpha^2\left(\frac{1}{c} - \frac{1}{b}\right) + \beta^2\left(\frac{1}{a} - \frac{1}{c}\right) + \gamma^2\left(\frac{1}{b} - \frac{1}{a}\right) = 0,$$

$$\frac{b-c}{a\alpha_1^2} + \frac{c-a}{b\beta_1^2} + \frac{a-b}{c\gamma_1^2} = 0.$$

38. There is only one point within a triangle, such that if perpendiculars be drawn from it to the sides, circles can be inscribed in each of the three resulting quadrilaterals; prove this, and if ρ_1, ρ_2, ρ_3 be the radii of these circles, and ρ that of the inscribed circle of the triangle, then

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)\left(\frac{1}{\rho_2} - \frac{1}{\rho}\right) + \left(\frac{1}{\rho_2} - \frac{1}{\rho}\right)\left(\frac{1}{\rho_3} - \frac{1}{\rho}\right) + \left(\frac{1}{\rho_3} - \frac{1}{\rho}\right)\left(\frac{1}{\rho_3} - \frac{1}{\rho}\right) = \frac{1}{\rho^2}.$$

39. A circle is inscribed in a plane triangle ABC. Another circle is described so as to touch the two sides AB, AC, and the last circle; again, a third circle is inscribed so as to touch the same two sides AB, AC, and the second circle, and so on. Circles are also inscribed in the same way so as to touch BC, BA and CA, CB. Shew that the area of the inscribed circle is to the sum of the areas of all the other circles as 1 is to

$$\sin^4\frac{B+C}{4}\operatorname{cosec}\frac{A}{2} + \sin^4\frac{C+A}{4}\operatorname{cosec}\frac{B}{2} + \sin^4\frac{A+B}{4}\operatorname{cosec}\frac{C}{2}.$$

40. O and O' are respectively the centres of the circles described about and inscribed in a plane triangle ABC. Join OA, OB, OC, $O'A$, $O'B$, $O'C$, and let R_a, R_b, R_c, r_a, r_b, r_c be respectively the radii of the circles circumscribing the triangles BOC, COA, AOB, $BO'C$, $CO'A$, $AO'B$. If R be the radius of the circle circumscribing the given triangle ABC, shew that

$$\frac{r_a r_b r_c}{abc} = \frac{R}{a+b+c}, \text{ and } \frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} = \frac{abc}{R^3}.$$

41. From any point P within or without a triangle ABC, perpendiculars PA', PB', PC' are dropped upon the sides BC, CA, AB; and circles are described about the triangles $PA'B'$, $PB'C'$, $PC'A'$. Shew that the area of the triangle formed by joining the centres of these circles is one-fourth of the area of the triangle ABC.

42. Three circles touch each other externally; prove that the square of the area of the triangle formed by joining their centres is equal to the product of the sum and product of their radii.

43. If the sides of a triangle be in geometrical progression, and the perpendiculars from the angles upon the opposite sides be taken as the sides of a new triangle, then the angles of this new triangle will be equal to those of the original triangle.

44. If α, β, γ be the ratios which the sides a, b, c of a triangle bear to the perpendiculars upon them from the opposite angles A, B, C, then $\alpha^2+\beta^2+\gamma^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)+4=0$.

45. In any triangle shew that

$$c=(a-b)\frac{\cos\frac{C}{2}}{\sin\frac{A-B}{2}}=(a+b)\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}.$$

46. The sides of a triangle are 65 and 25, and the difference of the opposite angles is 60°; find all the angles, having given

$\log 3=\cdot4771213$, $\log 2=\cdot3010300$,

$L\tan 52^\circ\,24'=10\cdot1134508$, $L\tan 52^\circ\,25'=10\cdot1137122$.

47. If perpendiculars be drawn from the angles of a triangle to the opposite sides, shew that the sides of the triangle formed by joining the feet of those perpendiculars are $a\cos A$, $b\cos B$, and $c\cos C$; and thence shew that

$$\frac{a^2\cos^2 A-b^2\cos^2 B-c^2\cos^2 C}{2bc\cos B\cos C}=\cos 2A.$$

48. Six circles are inscribed between the three escribed circles of a triangle and the angular points, each touching a side and a side produced; prove that the products of their radii taken alternately are equal.

49. If R be the radius of the circle circumscribing a triangle, ρ the radius of an escribed circle, the distance of the centres of these circles is $\sqrt{(R^2 + 2R\rho)}$.

50. Lines are drawn from the angles A, B, C of a triangle through any point P meeting the opposite sides of the triangle in the points A', B', C' respectively; shew that

$$AB' . BC' . CA' = AC' . BA' . CB'.$$

51. Shew that the perpendiculars from the angles of a triangle upon the opposite sides meet in a point.

52. Shew that the lines which bisect the internal angles of a triangle meet in a point.

53. Shew that the lines which join the angles of a triangle with the middle points of the opposite sides meet in a point.

54. Shew that the lines which join the angles of a triangle with the points where the inscribed circle touches the opposite sides respectively, meet in a point.

55. A quadrilateral figure is so taken that a circle can be described about it and inscribed in it. If its sides be produced in both directions, and r_a, r_b, r_c, r_d, be the radii of the circles, inscribed in the triangles formed on two sides, and escribed on the other two sides, then $r_a r_b r_c r_d = r^4$, where r is the radius of the circle inscribed in the quadrilateral.

XVII. ON THE USE OF SUBSIDIARY ANGLES IN SOLVING EQUATIONS AND IN ADAPTING FORMULÆ TO LOGARITHMIC COMPUTATION.

259. We shall now shew how to obtain the numerical values of the roots of a quadratic equation by the aid of Trigonometrical Tables.

(1) Suppose the equation to be

$$x^2 - 2px + q = 0,$$

where p and q are both positive; from this equation we obtain

$$x = p \pm \sqrt{(p^2 - q)} = p\left\{1 \pm \sqrt{\left(1 - \frac{q}{p^2}\right)}\right\}.$$

Now if q is less than p^2 assume $\frac{q}{p^2} = \sin^2\theta$; thus

$$x = p(1 \pm \cos\theta) = 2p\cos^2\frac{\theta}{2}, \text{ or } 2p\sin^2\frac{\theta}{2}.$$

If q is greater than p^2 the roots are impossible; we may then assume $\frac{q}{p^2} = \sec^2\theta$; thus

$$x = p\,\{1 \pm \sqrt{(-1)}\tan\theta\}.$$

(2) Suppose the equation to be

$$x^2 - 2px - q = 0,$$

where p and q are both positive; from this equation we obtain

$$x = p \pm \sqrt{(p^2 + q)} = p\left\{1 \pm \sqrt{\left(1 + \frac{q}{p^2}\right)}\right\}.$$

Now assume $\tan^2\theta = \frac{q}{p^2}$; thus

$$x = p\,(1 \pm \sec\theta) = p\frac{\cos\theta \pm 1}{\cos\theta} = \sqrt{q}\frac{\cos\theta \pm 1}{\sin\theta}$$

$$= \sqrt{q}\cot\frac{\theta}{2} \text{ or } -\sqrt{q}\tan\frac{\theta}{2}.$$

(3) If the equation is of the form $x^2+2px+q=0$, where p and q are positive, we can solve the equation $x^2-2px+q=0$, and then change the sign of the roots (Algebra, Art. 340).

(4) If the equation be of the form $x^2+2px-q=0$, where p and q are positive, we can solve the equation $x^2-2px-q=0$, and then change the sign of the roots.

260. In like manner we may obtain the numerical value of the roots of a cubic equation by the aid of Trigonometrical Tables; we will exemplify this by considering one case.

Let the equation be $x^3-qx-r=0$, and suppose $27r^2$ less than $4q^3$. Put $x=ny$; thus

$$n^3y^3-qny-r=0,$$

therefore $$y^3-\frac{qy}{n^2}-\frac{r}{n^3}=0.$$

Now by Art. 91, $\cos^3 a-\frac{3}{4}\cos a-\frac{\cos 3a}{4}=0$;

assume $y=\cos a$, $\frac{3}{4}=\frac{q}{n^2}$; then $\frac{r}{n^3}=\frac{\cos 3a}{4}$;

thus $$n=\left(\frac{4q}{3}\right)^{\frac{1}{2}},\quad \cos 3a=4r\left(\frac{3}{4q}\right)^{\frac{3}{2}};$$

the last equation determines $3a$, and thus a is known, then

$$y=\cos a \text{ and } x=n\cos a=\left(\frac{4q}{3}\right)^{\frac{1}{2}}\cos a.$$

The value of $\cos 3a$ is less than unity, since we have supposed $27r^2$ less than $4q^3$.

It appears from Art. 105 that we might also suppose

$$y=\cos\left(\frac{2\pi}{3}\pm a\right),$$

consistently with the value of $\cos 3\alpha$ given above; thus finally the three roots of the cubic equation are

$$2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\alpha \quad \text{and} \quad 2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}\pm\alpha\right),$$

where
$$\cos 3\alpha = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}.$$

261. "If in mathematical researches equations like those that have been given of the second and third degree, presented themselves to be solved, their solution would be conveniently effected by the preceding methods, and by the aid of the Trigonometrical Tables; but the truth is, in the application of Mathematics to Physics the solution of equations is an operation that very rarely is requisite, and consequently the preceding application of Trigonometrical Formulæ is to be considered as a matter rather of curiosity than of utility."—(Woodhouse's *Trigonometry.*)

262. To the examples which have already occurred of the use of subsidiary angles we will add two more.

(1) Required to adapt $a+b$ to logarithmic computation. If a and b are necessarily positive we may proceed thus; assume $\frac{b}{a} = \tan^2\theta$; then

$$a+b = a\left(1+\frac{b}{a}\right) = a(1+\tan^2\theta) = a\sec^2\theta.$$

If a and b are not necessarily both positive we may proceed thus; assume $\frac{b}{a} = \tan\theta$, then

$$a+b = a\left(1+\frac{b}{a}\right) = a(1+\tan\theta) = \frac{a\sqrt{2}}{\cos\theta}\left(\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}\right)$$

$$= \frac{a\sqrt{2}}{\cos\theta}\sin\left(\theta+\frac{\pi}{4}\right).$$

(2) Required to adapt $a\cos\alpha \pm b\sin\alpha$ to logarithmic computation. Let $\frac{b}{a} = \tan\theta$; thus

$$a\cos\alpha \pm b\sin\alpha = a\left(\cos\alpha \pm \frac{b}{a}\sin\alpha\right) = a(\cos\alpha \pm \tan\theta\sin\alpha)$$

$$= \frac{a}{\cos\theta}\cos(\alpha - \theta) \quad \text{or} \quad \frac{a}{\cos\theta}\cos(\alpha + \theta).$$

MISCELLANEOUS EXAMPLES.

1. Solve $x^3 + 9x^2 + 21x + 13 = 0$.

2. Shew that the roots of the equation $x^3 - 3x - 1 = 0$ are $2\cos 20^\circ$, $-2\sin 10^\circ$, $-2\cos 40^\circ$.

3. Shew that the roots of the equation $x^5 - px^3 + qx \pm r = 0$ are $2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\frac{\alpha}{5}$ and $2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\frac{3\pi \pm \pi \pm \alpha}{5}$, where $\cos^2\alpha = \frac{r^2}{4}\left(\frac{5}{p}\right)^5$ provided $p^2 = 5q$ and $\left(\frac{r}{2}\right)^2$ be less than $\left(\frac{p}{5}\right)^5$.

4. Find the roots of the equation

$$x^5 - 10x^3 + 20x - 8 = 0.$$

5. A person wishes to ascertain the side BC of a triangular field ABC, but is only able to make measurement of *lines* within the boundary of a circle which passes through A and touches BC; shew how after measuring four lines he may determine BC.

6. Two men standing at the same point C observe the horizontal angle subtended by two objects A and B; they then both move away, one in the direction AC, the other in the direction BC, until each observes the horizontal angle to be half what it was before. The distance each walked being given and the horizontal angle at C, determine the distance AB.

7. The altitude of a balloon at noon is observed at three places A, B, C simultaneously to be 45°, 45°, and 60° respectively; A and B are respectively west and north of C; form an equation for determining the height of the balloon.

8. The distances b and c of a station A from two other stations B and C are known, and the angle BAC is required. It not being practicable to observe the angle BAC, the angle BOC (α) and the angle AOC (β) are taken at a position O situated in the plane ABC, at a small known distance n from A. Shew that if θ be the circular measure of the angle $(BAC - BOC)$ then approximately

$$\theta = n\left\{\frac{\sin(\alpha-\beta)}{b} + \frac{\sin\beta}{c}\right\}.$$

9. At a distance of 50 feet from the foot of a tower the elevation of its top is 45°; if the elevation and the distance be correctly measured within $1'$ and 1 inch respectively, find approximately the greatest error in the height.

10. A person standing at a distance a from a tower surmounted by a spire, observes the tower and spire to subtend the same angle; if b be the known height of the tower, express the height of the spire (c) in terms of b and a.

If γ be the error in the height of the spire corresponding to a small error β in the height of the tower, shew that

$$\frac{\gamma}{c} = \frac{\beta}{b}\left\{1 + \frac{4b^2a^2}{a^4 - b^4}\right\}.$$

11. One side of a triangle and the opposite angle remain constant; shew that the small variations of the other sides γ and β are connected by the relation

$$\gamma \sec C + \beta \sec B = 0.$$

12. The angular altitude and breadth of a cylindrical tower on a level plane are observed to be α and β respectively; and at a point a feet nearer the tower they are found to be α' and β'; find

the height and radius of the tower. Find also the relation existing between α, α', β, β'.

13. In the preceding question if the observed angular breadth be subject to an error δ, and if ρ be the greatest consequent error in the calculated radius (r), shew that ρ will be given by the equation

$$\frac{2\rho}{r} = \cot\frac{1}{4}(\beta' - \beta)\left\{\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\beta'}{2} - \cot\frac{\beta}{2}\cot\frac{\beta'}{2}\right\}\delta.$$

If $\beta = 60°$, $\beta' = 120°$, $\delta = 6'$, find approximately the ratio of the greatest error in the calculated radius to the radius.

14. P, Q, R are three known positions in a straight line, and PQ, QR are observed to subtend equal angles at a certain point S; find the error in the calculated distance of S from Q in consequence of a small error α in the observed angles.

XVIII. INVERSE TRIGONOMETRICAL FUNCTIONS.

263. The equation $\sin x = a$ asserts that x is an angle of which the sine is a; it is found convenient to have a notation for expressing this relation in which x stands alone. The notation used is this, $x = \sin^{-1}a$. Similarly the equation $x = \cos^{-1}a$ expresses that x is an angle of which the cosine is a; and $x = \tan^{-1}a$ expresses that x is an angle of which the tangent is a; and so on.

264. Experience will prove that the notation here given is often convenient; and we may shew that it is not altogether an *arbitrary* notation, but one that naturally presents itself. For, let any function of x be denoted by $f(x)$; then the same function of $f(x)$, that is, $f\{f(x)\}$, may be briefly and conveniently denoted by $f^2(x)$. Thus, for example, the logarithm of the logarithm of x may be denoted by $\log^2 x$. Similarly $f[f\{f(x)\}]$ may be briefly and conveniently denoted by $f^3(x)$; and so on. Thus with this notation we have, when m and n are positive integers,

$$f^m f^n(x) = f^{m+n}(x).$$

Now we may examine what meaning it will be necessary to ascribe to $f^0(x)$, in order that the relation just given may hold when m or n is zero. Suppose $n = 0$, then the relation becomes

$$f^m f^0(x) = f^m(x),$$

this leads us to settle that $f^0(x)$ shall be considered equal to x.

Again we may examine what meaning it will be necessary to ascribe to $f^{-1}(x)$ in order that the relation $f^m f^n(x) = f^{m+n}(x)$ may hold when m or n is -1. Suppose $m = 1$ and $n = -1$; thus the relation becomes

$$f f^{-1}(x) = f^0(x) = x,$$

so that $f^{-1}(x)$ must denote a quantity whose function f is x.

Thus $\sin^{-1}x$ should denote a quantity whose sine is x; and this is the meaning which we have already assigned to the symbol.

It will be observed that consistently with the remarks here made, $\sin^2 x$ should stand for $\sin(\sin x)$, and not for $\sin x \times \sin x$. But as $\sin(\sin x)$ is a function which rarely occurs, it is customary to use $\sin^2 x$ for what should be denoted by $(\sin x)^2$.

265. Any relation which has been established among trigonometrical functions may be expressed by means of the *inverse* notation. Thus, for example, we know that

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta};$$

this may be written

$$2\theta = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right);$$

let $\tan\theta = a$, so that $\theta = \tan^{-1}a$; thus

$$2\tan^{-1}a = \tan^{-1}\frac{2a}{1-a^2}.$$

Similarly the relation $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ may be expressed thus,

$$3\sin^{-1}a = \sin^{-1}(3a - 4a^3).$$

EXAMPLES.

1. Prove that $\tan^{-1}\frac{3}{4} = 2\tan^{-1}\frac{1}{3}$.

2. Find the value of $\sin(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2})$.

3. Prove that $\sin^{-1}\frac{77}{85} = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$.

4. Find the value of $\tan(\tan^{-1} x + \cot^{-1} x)$.

5. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

6. Prove that $\tan^{-1} a = \tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1} c$.

7. Find the tangent of

$$3\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{26} - \frac{\pi}{4}.$$

8. Shew that

$$\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\} - \tan^{-1}\{(\sqrt{2}-1)\tan\alpha\} = \tan^{-1}(\sin 2\alpha).$$

9. If $\tan(\theta-\alpha)\tan(\theta-\beta) = \tan^2\theta$; then

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

10. Prove that $\cos^{-1}\frac{9}{\sqrt{(82)}} + \operatorname{cosec}^{-1}\frac{\sqrt{(41)}}{4} = \frac{\pi}{4}$.

11. Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$.

12. Prove that $3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} = \frac{\pi}{4} - \tan^{-1}\frac{1}{1985}$.

13. Prove that $\tan^{-1}\frac{2a-b}{b\sqrt{3}} + \tan^{-1}\frac{2b-a}{a\sqrt{3}} = \frac{\pi}{3}$.

14. Prove that $\tan(2\tan^{-1} a) = 2\tan(\tan^{-1} a + \tan^{-1} a^3)$.

15. Prove that

$$\tan^{-1}(\tfrac{1}{2}\tan 2A)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0.$$

16. Prove that

$$\frac{2b}{a}=\tan\left(\frac{\pi}{4}+\tfrac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\tfrac{1}{2}\cos^{-1}\frac{a}{b}\right).$$

17. Prove that

$$\frac{a^3}{2}\operatorname{cosec}^2\left(\tfrac{1}{2}\tan^{-1}\frac{a}{b}\right)+\frac{b^3}{2}\sec^2\left(\tfrac{1}{2}\tan^{-1}\frac{b}{a}\right)=(a+b)(a^2+b^2).$$

Solve the following seven equations in x.

18. $\sin^{-1}x+\sin^{-1}\dfrac{x}{2}=\dfrac{\pi}{4}.$

19. $\sin^{-1}\dfrac{2a}{1+a^2}+\sin^{-1}\dfrac{2b}{1+b^2}=2\tan^{-1}x.$

20. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x.$

21. $\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x.$

22. $\tan^{-1}\tfrac{1}{4}+2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{6}+\tan^{-1}\dfrac{1}{x}=\dfrac{\pi}{4}.$

23. $\sin 2\cos^{-1}\cot 2\tan^{-1}x=0.$

24. $\tan^{-1}\dfrac{1}{a-1}=\tan^{-1}\dfrac{1}{x}+\tan^{-1}\dfrac{1}{a^2-x+1}.$

25. If $\sec\theta-\operatorname{cosec}\theta=\dfrac{4}{3}$, shew that $\theta=\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}.$

26. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, shew that $\theta=\pm\tfrac{1}{2}\sin^{-1}\tfrac{3}{4}.$

27. Shew that if $\sin^2\theta+\sin^2\phi=\tfrac{1}{2}$, one of the values of ψ which satisfy the equation

$$\psi=\sin^{-1}(\sin\theta+\sin\phi)+\sin^{-1}(\sin\theta-\sin\phi)$$

is

$$(2n+1)\frac{\pi}{2}.$$

28. Find x from the following equation,

$$3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}.$$

29. Shew that one of the expressions

$$\sin^{-1}\frac{2b+a-c}{a+c}\pm 2\sin^{-1}\sqrt{\left(\frac{a+b}{a+c}\right)}$$

is an odd multiple of $\frac{\pi}{2}$.

30. Find all the positive integral solutions of

$$\tan^{-1}x+\tan^{-1}\frac{1}{y}=\tan^{-1}3.$$

31. Shew that if c be a positive integer, the equation

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}c$$

has no positive integral solutions; while the equation

$$\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{y}=\tan^{-1}\frac{1}{c}$$

has as many as there are different divisors of $1+c^2$.

32. Prove that $\tan^{-1}\frac{x}{y}=\tan^{-1}\frac{c_1x-y}{c_1y+x}+\tan^{-1}\frac{c_2-c_1}{c_2c_1+1}$

$$+\tan^{-1}\frac{c_3-c_2}{c_3c_2+1}+\ldots\ldots+\tan^{-1}\frac{c_n-c_{n-1}}{c_nc_{n-1}+1}+\tan^{-1}\frac{1}{c_n},$$

where c_1, c_2, ... c_n are any quantities whatever.

33. The sum of any number of angles

$$\sin^{-1}\frac{2ab}{a^2+b^2},\quad \sin^{-1}\frac{2a'b'}{a'^2+b'^2},\ \ldots\ldots$$

may be expressed in the form

$$\sin^{-1}\frac{2mn}{m^2+n^2},$$

where m and n are *rational* functions of a, b, a', b', ...

34. Write down the general value of $\sin^{-1}\frac{(-1)^m}{2}$, where m is an integer.

35. Write down the general value of $\cos^{-1}\frac{(-1)^m}{2}$, where m is an integer.

XIX. DE MOIVRE'S THEOREM.

266. The student has already learned from Algebra that although the square root of a negative quantity is the symbol of an impossible operation, yet such roots are of use in mathematical investigations. It is usual to adopt the convention that

$$\sqrt{(-a^2)} = a\sqrt{(-1)},$$

and that such expressions as $a\sqrt{(-1)}$ shall be subject to all the laws of algebraical transformations. In the remainder of the present work it will be found that $\sqrt{(-1)}$ occurs very frequently in our investigations; we shall for the present assume that this expression may be freely used like any real algebraical expression, and hereafter we shall give some remarks on the question of the validity of demonstrations which are obtained by the use of the symbol $\sqrt{(-1)}$. (See also *Algebra*, Chap. xxv.)

267. *De Moivre's Theorem. Whatever be the value of* n *positive or negative, integral or fractional,* $\cos n\theta + \sqrt{(-1)}\sin n\theta$ *is one of the values of* $\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n$.

Multiply $\cos\alpha + \sqrt{(-1)}\sin\alpha$ by $\cos\beta + \sqrt{(-1)}\sin\beta$; the product is

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \sqrt{(-1)}\{\sin\alpha\cos\beta + \cos\alpha\sin\beta\},$$

that is, $\qquad \cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)$;

multiply the last expression by

$$\cos\gamma + \sqrt{(-1)}\sin\gamma;$$

the product is

$$\cos(\alpha+\beta+\gamma) + \sqrt{(-1)}\sin(\alpha+\beta+\gamma).$$

By proceeding in this way we obtain the product of any number of factors of the form $\cos\alpha + \sqrt{(-1)}\sin\alpha$. Suppose there are n of these factors, each factor being $\cos\theta + \sqrt{(-1)}\sin\theta$; we then have

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n = \cos n\theta + \sqrt{(-1)}\sin n\theta.$$

This proves De Moivre's theorem when n is a *positive integer.*

Next, let n be a *negative integer;* suppose $n = -m$, then

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^{-m}$$

$$= \frac{1}{\{\cos\theta + \sqrt{(-1)}\sin\theta\}^m}$$

$$= \frac{1}{\cos m\theta + \sqrt{(-1)}\sin m\theta};$$

multiply both numerator and denominator by

$$\cos m\theta - \sqrt{(-1)}\sin m\theta,$$

thus we obtain $$\frac{\cos m\theta - \sqrt{(-1)}\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta};$$

that is $$\cos m\theta - \sqrt{(-1)}\sin m\theta;$$

that is $$\cos(-m\theta) + \sqrt{(-1)}\sin(-m\theta),$$

or $$\cos n\theta + \sqrt{(-1)}\sin n\theta.$$

This proves De Moivre's theorem when n is a *negative integer.*

Thus, since when n is any integer,

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n = \cos n\theta + \sqrt{(-1)}\sin n\theta,$$

it follows that $\cos\theta + \sqrt{(-1)}\sin\theta$ is one of the values of

$$\{\cos n\theta + \sqrt{(-1)}\sin n\theta\}^{\frac{1}{n}},$$

when n is any integer.

Lastly, then, let n be a fraction; suppose $n = \frac{p}{q}$, then

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{p}{q}}$$

$$= \{\cos p\theta + \sqrt{(-1)}\sin p\theta\}^{\frac{1}{q}},$$

and one of the values of the last expression is

$$\cos\frac{p\theta}{q} + \sqrt{(-1)}\sin\frac{p\theta}{q}.$$

Thus De Moivre's theorem is completely established.

268. We have shewn that when n is fractional,

$$\cos n\theta + \sqrt{(-1)}\sin n\theta$$

is *one* of the values of

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n;$$

we shall now shew how *all* the values of the last expression may be obtained. Suppose $n = \frac{p}{q}$. Now $\cos\theta$ and $\sin\theta$ remain unchanged when θ is increased by any multiple of 2π, while by putting $\theta + 2r\pi$ instead of θ, and ascribing to r in succession different integral values the expression $\cos n\theta + \sqrt{(-1)}\sin n\theta$, assumes q different values and no more. For suppose r successively equal to $0, 1, 2, \ldots\ldots q-1$; then we obtain the series of angles

$$\frac{p\theta}{q}, \frac{p(\theta + 2\pi)}{q}, \frac{p(\theta + 4\pi)}{q}, \ldots\ldots \frac{p(\theta + 2q\pi - 2\pi)}{q},$$

and we know that no two of these angles can have the same sine and the same cosine, because no two of these angles are equal or differ by a multiple of 2π. (See Art. 93.) Hence we obtain q different values of the expression $\cos n\theta + \sqrt{(-1)}\sin n\theta$. We shall not in this way obtain *more* than q different values, for if $r = s + mq$, where m is *any integer positive or negative*,

$$\cos n(\theta + 2r\pi) \text{ and } \sin n(\theta + 2r\pi)$$

are respectively equal to

$$\cos n(\theta + 2s\pi) \text{ and } \sin n(\theta + 2s\pi).$$

We can thus find q different values for the expression

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{p}{q}};$$

that is, we can find q different expressions, which by being raised to the q^{th} power, produce $\cos p\theta + \sqrt{(-1)}\sin p\theta$. And it is known from the theory of equations that there are q values of x, and no more, which satisfy the equation $x^q = c$, where c is either real or of the form $a + b\sqrt{(-1)}$; thus we infer that we know *all* the values of the expression

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{p}{q}}.$$

269. We proceed to deduce some important results from De Moivre's theorem. In the equation

$$\cos n\theta + \sqrt{(-1)}\sin n\theta = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^n,$$

suppose n a positive integer. Expand the right-hand member by the Binomial Theorem, and equate the possible and impossible parts of the two members; thus

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1 \cdot 2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta\sin^4\theta - \ldots\ldots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\cos^{n-5}\theta\sin^5\theta - \ldots\ldots$$

270. The preceding formulæ hold whether n be odd or even, but the last terms of the expressions on the right-hand side are different in the two cases, and it will be useful to distinguish the cases.

If n be even, the last term of the expansion of

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n$$

is possible, namely, $(-1)^{\frac{n}{2}}\sin^n\theta$; and the last term but one is impossible, namely, $n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta$, which may be written $\sqrt{(-1)}\,n(-1)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta$. Thus when n is even

the last term of $\cos n\theta$ is $(-1)^{\frac{n}{2}}\sin^n\theta$,

and the last term of $\sin n\theta$ is $n(-1)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta$.

If n be odd, the last term of the expansion of $\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n$ is impossible, namely $(-1)^{\frac{n}{2}}\sin^n\theta$, which may be written

$$\sqrt{(-1)}(-1)^{\frac{n-1}{2}}\sin^n\theta;$$

and the last term but one is possible, namely

$$n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta.$$

Thus, when n is odd,

the last term of $\cos n\theta$ is $n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta$,

and the last term of $\sin n\theta$ is $(-1)^{\frac{n-1}{2}}\sin^n\theta$.

271. From the formulæ for $\sin n\theta$ and $\cos n\theta$ we can deduce an expression for $\tan n\theta$ in terms of the powers of $\tan\theta$.

For $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$

$$= \frac{n\cos^{n-1}\theta\sin\theta - \dfrac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \ldots}{\cos^n\theta - \dfrac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\sin^2\theta + \ldots}.$$

Now divide both numerator and denominator of this expression by $\cos^n \theta$; thus we find for $\tan n\theta$ the expression

$$\frac{n\tan\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\tan^5\theta - \dots}{1 - \frac{n(n-1)}{1\,.\,2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta - \dots}.$$

272. If n be even, the last term of the numerator of $\tan n\theta$ is $n(-1)^{\frac{n-2}{2}}\tan^{n-1}\theta$, and the last term of the denominator is $(-1)^{\frac{n}{2}}\tan^n\theta$. If n be odd, the last term of the numerator is $(-1)^{\frac{n-1}{2}}\tan^n\theta$, and the last term of the denominator is $n(-1)^{\frac{n-1}{2}}\tan^{n-1}\theta$.

These results follow from those established in Art. 270.

273. We may also obtain general formulæ for the sine, cosine, and tangent of the sum of any number of angles which are not all equal. We have seen (Art. 267) that

$$\{\cos\alpha + \sqrt{(-1)}\sin\alpha\}\{\cos\beta + \sqrt{(-1)}\sin\beta\}\{\cos\gamma + \sqrt{(-1)}\sin\gamma\}\dots\dots$$
$$= \cos(\alpha+\beta+\gamma+\dots\dots) + \sqrt{(-1)}\sin(\alpha+\beta+\gamma\dots\dots).$$

Now $\cos\alpha + \sqrt{(-1)}\sin\alpha = \cos\alpha\,\{1 + \sqrt{(-1)}\tan\alpha\}$,

$$\cos\beta + \sqrt{(-1)}\sin\beta = \cos\beta\,\{1 + \sqrt{(-1)}\tan\beta\},$$

....

thus we obtain

$$\cos\alpha\cos\beta\cos\gamma\dots\{1+\sqrt{(-1)}\tan\alpha\}\{1+\sqrt{(-1)}\tan\beta\}\{1+\sqrt{(-1)}\tan\gamma\}\dots$$
$$= \cos(\alpha+\beta+\gamma+\dots\dots) + \sqrt{(-1)}\sin(\alpha+\beta+\gamma+\dots\dots)$$

Let s_1 denote the sum $\tan\alpha + \tan\beta + \tan\gamma + \dots\dots$; let s_2 denote the sum of the products of the tangents taken two at a time; let s_3 denote the sum of the products of the tangents taken three at a time; and so on.

Then by multiplying together the factors $1+\sqrt{(-1)}\tan\alpha$, $+\sqrt{(-1)}\tan\beta$, $1+\sqrt{(-1)}\tan\gamma$, ... and equating possible and impossible parts we obtain

$$\cos(\alpha+\beta+\gamma+\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots\{1-s_2+s_4-s_6+\ldots\},$$

$$\sin(\alpha+\beta+\gamma+\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots\{s_1-s_3+s_5-s_7+\ldots\}.$$

By division,

$$\tan(\alpha+\beta+\gamma+\ldots)=\frac{s_1-s_3+s_5-s_7+\ldots}{1-s_2+s_4-s_6+\ldots}.$$

If n be even, the last term in the numerator is $(-1)^{\frac{n-2}{2}}s_{n-1}$, and the last term in the denominator is $(-1)^{\frac{n}{2}}s_n$; if n be odd, the last term in the numerator is $(-1)^{\frac{n-1}{2}}s_n$, and the last term in the denominator is $(-1)^{\frac{n-1}{2}}s_{n-1}$. If the angles $\alpha, \beta, \ldots$ are all equal, the formula will coincide with that given in Art. 271.

274. We shall now prove formulæ for the expansion of $\sin\alpha$ and $\cos\alpha$ in series of powers of α.

We have, when n is a positive integer,

$$\cos n\theta=\cos^n\theta-\frac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^4\theta-\ldots\ldots$$

Let $n\theta=\alpha$; and suppose n to increase without limit, and let θ so change that n may remain a positive integer and $n\theta$ be always equal to α; thus θ must diminish without limit. The preceding equation may be written

$$\cos\alpha=\cos^n\theta-\frac{\alpha(\alpha-\theta)}{1\,.\,2}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{\underline{|4}}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4-\ldots\ldots$$

Now when n increases without limit, and, therefore, θ diminishes without limit, $\frac{\sin\theta}{\theta}$ is equal to unity, and so is every power of $\frac{\sin\theta}{\theta}$ up to $\left(\frac{\sin\theta}{\theta}\right)^n$; also $\cos\theta$ is unity and so is every power of $\cos\theta$ up to $\cos^n\theta$ (Art. 150). Hence the above formula becomes

$$\cos\alpha = 1 - \frac{\alpha^2}{1\,.\,2} + \frac{\alpha^4}{\lfloor 4} - \frac{\alpha^6}{\lfloor 6} + \ldots\ldots$$

Also

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots$$

thus $\sin\alpha = \alpha\cos^{n-1}\theta\frac{\sin\theta}{\theta} - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{\lfloor 3}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \ldots\ldots$

Hence, by supposing n to increase without limit, we obtain

$$\sin\alpha = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \ldots\ldots$$

The results of this article are of the greatest importance; we shall make some remarks upon them in the next three articles.

275. It must be observed with respect to the formulæ established for the expansion of sin α and cos α, that *α is the circular measure of the angle* considered; for it is only when an angle is estimated in circular measure that $\frac{\sin\theta}{\theta}$ is unity when θ is indefinitely diminished. It is easy to obtain the requisite modification of the formulæ when any other unit of angular measurement is adopted. Thus, for example,

$$\sin n^\circ = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \ldots\ldots$$

where α is the circular measure of the angle of n°; thus $\alpha = \frac{n\pi}{180}$, and we have

$$\sin n^0 = \frac{n\pi}{180} - \frac{1}{\underline{|3}}\left(\frac{n\pi}{180}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{n\pi}{180}\right)^5 - \ldots\ldots$$

Similarly $\cos n^0 = 1 - \frac{1}{2}\left(\frac{n\pi}{180}\right)^2 + \frac{1}{\underline{|4}}\left(\frac{n\pi}{180}\right)^4 - \ldots\ldots$

276. *The series for* sin a *and* cos a *are convergent for all values of* a.

The n^{th} term in the series for $\sin a$ is $\frac{(-1)^{n-1} a^{2n-1}}{\underline{|2n-1}}$; hence the numerical value of the ratio of the $(n+1)^{\text{th}}$ term to the n^{th} is $\frac{a^2}{2n(2n+1)}$; and whatever be the value of a we can take n so large that for such value of n and all greater values $\frac{a^2}{2n(2n+1)}$ shall be less than any assigned quantity; hence the series is convergent (*Algebra*, Art. 559).

Similarly it may be shewn that the series for $\cos a$ is always convergent.

277. The proof given in Art. 274 involves one point that may not at first appear quite satisfactory. The $(r+1)^{\text{th}}$ term of $\cos a$ is strictly

$$(-1)^r \frac{n(n-1)(n-2)\ldots(n-2r+1)}{\underline{|2r}} \cos^{n-2r}\theta \sin^{2r}\theta;$$

this we write in the form

$$(-1)^r \frac{a(a-\theta)(a-2\theta)\ldots(a-2r\theta+\theta)}{\underline{|2r}} \cos^{n-2r}\theta \left(\frac{\sin\theta}{\theta}\right)^{2r}.$$

Now it is proved in Art. 150 that the limit of $\cos^{n-2r}\theta$ is unity, and also that the limit of $\left(\frac{\sin\theta}{\theta}\right)^{2r}$ is unity; the only question is whether the limit of

$$\frac{a(a-\theta)(a-2\theta)\ldots(a-2r\theta+\theta)}{\underline{|2r}} \text{ is } \frac{a^{2r}}{\underline{|2r}}$$

for all values of r. This is obviously true when $r=1$; that is, the limit of $\frac{\alpha(\alpha-\theta)}{2}$ is $\frac{\alpha^2}{2}$; and we can shew by induction that the required result is always true. For assume that

$$\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots(\alpha-2r\theta+\theta)}{\lfloor 2r} = \frac{\alpha^{2r}}{\lfloor 2r} + R$$

where R diminishes without limit when θ does so, so that the limit of the right-hand member is $\frac{\alpha^{2r}}{\lfloor 2r}$; introduce a new factor $\frac{\alpha-2r\theta}{2r+1}$; thus

$$\frac{\alpha(\alpha-\theta)\ldots(\alpha-2r\theta)}{\lfloor 2r+1} = \left\{\frac{\alpha^{2r}}{\lfloor 2r} + R\right\}\left\{\frac{\alpha}{2r+1} - \frac{2r\theta}{2r+1}\right\}$$

$$= \frac{\alpha^{2r+1}}{\lfloor 2r+1} + \frac{R\alpha}{2r+1} - \frac{2r\theta}{2r+1}\left\{\frac{\alpha^{2r}}{\lfloor 2r} + R\right\};$$

and when θ diminishes without limit all the terms on the right-hand side vanish except $\frac{\alpha^{2r+1}}{\lfloor 2r+1}$, which is therefore the limit of the left-hand member. Similarly we can shew that when another factor $\frac{\alpha-2r\theta-\theta}{2r+2}$ is introduced the limit is $\frac{\alpha^{2r+2}}{\lfloor 2r+2}$; and so on.

278. The following example will shew how the series for $\cos\theta$ may be practically useful. Suppose two sides a and b of a triangle are known, and the included angle C; if C be a very *obtuse* angle we can give a convenient expression for the third side of the triangle.

For suppose $\pi-\theta$ to be the circular measure of the angle C, so that θ is very small; thus

$$c^2 = a^2 + b^2 - 2ab\cos C = a^2 + b^2 + 2ab\cos\theta$$

$$= a^2 + b^2 + 2ab\left(1 - \frac{\theta^2}{2}\right) \text{ approximately,}$$

$$= (a+b)^2 - ab\theta^2$$

$$= (a+b)^2\left\{1 - \frac{ab\theta^2}{(a+b)^2}\right\}.$$

Hence, by extracting the square root,

$$c=(a+b)\left\{1-\frac{ab\theta^2}{2(a+b)^2}\right\}\text{ approximately.}$$

EXAMPLES.

1. Extract the square root of $\cos 4A \pm \sqrt{(-1)}\sin 4A$.

2. Find the values of $(-1)^{\frac{1}{3}}$.

3. Obtain the six values of $(-1)^{\frac{1}{6}}$.

4. Find the three values of $\{1+\sqrt{(-1)}\}^{\frac{1}{3}}$.

5. Given $\dfrac{\sin\theta}{\theta}=\dfrac{2165}{2166}$, shew that θ is nearly the circular measure of 3°.

6. Given $\sin\left(\dfrac{\pi}{6}+\theta\right)=\cdot 51$, find approximately the value of θ, neglecting powers of θ above the second.

7. If
$$\tan x = x + \frac{a_3 x^3}{\lfloor 3} + \frac{a_5 x^5}{\lfloor 5} + \ldots\ldots$$
shew that
$$a_{2n+1}=\frac{(2n+1)\,2n}{1\,.\,2}a_{2n-1}-\frac{(2n+1)\,2n\,(2n-1)\,(2n-2)}{\lfloor 4}a_{2n-3}+\ldots$$

8. If
$$\theta\cot\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots$$
shew that
$$a_{2n}=\frac{a_{2n-2}}{\lfloor 3}-\frac{a_{2n-4}}{\lfloor 5}+\ldots+\frac{(-1)^{n-1}a_0}{\lfloor 2n+1}+\frac{(-1)^n}{\lfloor 2n};$$
hence find $\theta\cot\theta$ to four terms.

9. If
$$\sec\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots + a_{2n}\theta^{2n} + \ldots$$
shew that
$$a_{2n}=\frac{a_{2n-2}}{\lfloor 2}-\frac{a_{2n-4}}{\lfloor 4}+\ldots+\frac{(-1)^{n-1}a_0}{\lfloor 2n}.$$

10. If $\cos 2\alpha + \sqrt{(-1)} \sin 2\alpha$ be substituted for a in the expression $\dfrac{bc}{(a+b)(a+c)}$, and similar quantities for b and c, and the result reduced to the form $A + B\sqrt{(-1)}$, find the values of A and B in terms of α, β, γ.

11. Shew that

$$\{\cos\theta + \cos\phi + \sqrt{(-1)}(\sin\theta + \sin\phi)\}^n$$
$$+ \{\cos\theta + \cos\phi - \sqrt{(-1)}(\sin\theta + \sin\phi)\}^n$$
$$= 2^{n+1}\left(\cos\frac{\theta-\phi}{2}\right)^n \cos\frac{n(\theta+\phi)}{2}.$$

12. Shew that if $x = e^{\theta\sqrt{(-1)}}$, and $\sqrt{(1-c^2)} = nc - 1$,

$$1 + c\cos\theta = \frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right).$$

13. Prove the following rule for finding the length of a small circular arc; from eight times the chord of half the arc subtract the chord of the whole arc, and one-third of the remainder will give the length of the arc nearly.

14. From the identical equation

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1,$$

deduce the following by assuming

$$x = \cos 2\theta + \sqrt{(-1)}\sin 2\theta,$$

and corresponding assumptions for a, b, and c;

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)$$
$$+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\sin 2(\theta-\beta)$$
$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\sin 2(\theta-\gamma) = 0.$$

XX. EXPANSIONS OF SOME TRIGONOMETRICAL FUNCTIONS.

279. Let x denote $\cos\theta + \sqrt{(-1)}\sin\theta$; then

$$\frac{1}{x} = \frac{1}{\cos\theta + \sqrt{(-1)}\sin\theta} = \cos\theta - \sqrt{(-1)}\sin\theta;$$

thus $$x + \frac{1}{x} = 2\cos\theta, \quad \text{and } x - \frac{1}{x} = 2\sqrt{(-1)}\sin\theta;$$

also $$x^n = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^n = \cos n\theta + \sqrt{(-1)}\sin n\theta,$$

$$\frac{1}{x^n} = \frac{1}{\{\cos\theta + \sqrt{(-1)}\sin\theta\}^n} = \frac{1}{\cos n\theta + \sqrt{(-1)}\sin n\theta}$$
$$= \cos n\theta - \sqrt{(-1)}\sin n\theta;$$

thus $$x^n + \frac{1}{x^n} = 2\cos n\theta, \text{ and } x^n - \frac{1}{x^n} = 2\sqrt{(-1)}\sin n\theta.$$

We shall find this notation useful in the following investigations.

280. *To express* $\cos^n\theta$ *in terms of cosines of multiples of* θ *when* n *is a positive integer.*

$$2^n\cos^n\theta = \left(x + \frac{1}{x}\right)^n = x^n + nx^{n-1}.\frac{1}{x} + \frac{n(n-1)}{1.2}x^{n-2}.\frac{1}{x^2} + \dots$$
$$+ \frac{n(n-1)}{1.2}x^2.\frac{1}{x^{n-2}} + nx.\frac{1}{x^{n-1}} + \frac{1}{x^n}.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1.2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots;$$

but $$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^{n-2} + \frac{1}{x^{n-2}} = 2\cos(n-2)\theta, \text{ and so on};$$

therefore

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta + \dots$$
$$+\frac{n(n-1)\dots(n-r+1)}{\lfloor r}\cos(n-2r)\theta + \dots$$

The last term of the series on the right-hand side will take different forms according as n is even or odd. In the expansion of $\left(x+\frac{1}{x}\right)^n$ by the binomial theorem there are $n+1$ terms; thus when n is even, there will be a middle term, namely the $\left(\frac{n}{2}+1\right)^{\text{th}}$, which is

$$\frac{n(n-1)\dots(n-\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n} x^{\frac{n}{2}} \cdot \frac{1}{x^{\frac{n}{2}}}; \text{ that is, } \frac{n(n-1)\dots(\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n}.$$

Hence, when n is even, the last term of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\dots(\frac{1}{2}n+1)}{2\lfloor \frac{1}{2}n}.$$

When n is odd suppose it $=2m+1$; there are two middle terms in the expansion of $\left(x+\frac{1}{x}\right)^n$, namely, the $(m+1)^{\text{th}}$ and $(m+2)^{\text{th}}$; their sum is

$$\frac{n(n-1)\dots(n-m+1)}{\lfloor m}\left(x+\frac{1}{x}\right).$$

Hence when n is odd, the last term of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\dots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}\cos\theta.$$

281. We shall find that $\sin^n\theta$ can be expressed in terms of cosines of multiples of θ if n be an even positive integer, and in terms of sines of multiples of θ if n be an odd positive integer; this will appear in the following two articles.

282. *To express* $\sin^n\theta$ *in terms of cosines of multiples of* θ, *when* n *is an even positive integer.*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta=\left(x-\frac{1}{x}\right)^n=x^n-nx^{n-1}\cdot\frac{1}{x}+\frac{n(n-1)}{1\,.\,2}x^{n-2}\cdot\frac{1}{x^2}+\dots$$

$$+\frac{n(n-1)}{1\,.\,2}x^2\cdot\left(-\frac{1}{x}\right)^{n-2}+nx\left(-\frac{1}{x}\right)^{n-1}+\left(-\frac{1}{x}\right)^n.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n+\frac{1}{x^n}-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+\frac{n(n-1)}{1\,.\,2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)-\dots$$

$$+(-1)^{\frac{n}{2}}\frac{n(n-1)\dots(\frac{1}{2}n+1)}{\lfloor\frac{1}{2}n}.$$

Therefore

$$2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{1\,.\,2}(\cos n-4)\theta-\dots$$

$$+(-1)^{n-r}\frac{n(n-1)\dots(n-r+1)}{\lfloor r}\cos(n-2r)\theta+\dots$$

$$+(-1)^{\frac{n}{2}}\frac{n(n-1)\dots(\frac{1}{2}n+1)}{2\lfloor\frac{1}{2}n}.$$

283. *To express* $\sin^n\theta$ *in terms of sines of multiples of* θ *when* n *is an odd positive integer.*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta=\left(x-\frac{1}{x}\right)^n=x^n-nx^{n-1}\cdot\frac{1}{x}+\frac{n(n-1)}{1\,.\,2}x^{n-2}\cdot\frac{1}{x^2}$$

$$-\dots-\frac{n(n-1)}{1\,.\,2}x^2\cdot\frac{1}{x^{n-2}}+nx\cdot\frac{1}{x^{n-1}}-\frac{1}{x^n}.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n - \frac{1}{x^n} - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1 \,.\, 2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \dots$$

$$+ (-1)^{\frac{n-1}{2}} \frac{n(n-1)\dots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)} \left(x - \frac{1}{x}\right);$$

but
$$x^n - \frac{1}{x^n} = 2\sqrt{(-1)}\sin n\theta,$$

$$x^{n-2} - \frac{1}{x^{n-2}} = 2\sqrt{(-1)}\sin(n-2)\theta,$$

and so on; therefore

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1 \,.\, 2}\sin(n-4)\theta$$

$$-\frac{n(n-1)(n-2)}{\lfloor 3}\sin(n-6)\theta + \dots + (-1)^{\frac{n-1}{2}}\frac{n(n-1)\dots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\sin\theta.$$

284. If n be not a positive integer, the expressions for $\cos^n\theta$ and $\sin^n\theta$ in terms of the cosines and sines of multiples of θ are very complicated. For these we may refer to Peacock's *Algebra*, Vol. II. pp. 435—440.

285. In Art. 269 it is shewn that when n is a positive integer,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1 \,.\, 2}\cos^{n-2}\theta\sin^2\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta\sin^4\theta - \dots;$$

since $\sin^2\theta = 1 - \cos^2\theta,\quad \sin^4\theta = (1-\cos^2\theta)^2,$

and so on, it is obvious that $\cos n\theta$ can be expressed in terms of powers of $\cos\theta$; we will now give a direct investigation of this expression.

286. *To express* $\cos n\theta$ *in a series of descending powers of* $\cos\theta$ *when* n *is a positive integer.*

Let
$$x = \cos\theta + \sqrt{(-1)}\sin\theta,$$

so that $$x + \frac{1}{x} = 2\cos\theta, \text{ and } x^n + \frac{1}{x^n} = 2\cos n\theta;$$

now $$(1 - zx)\left(1 - \frac{z}{x}\right) = 1 - z\left(x + \frac{1}{x}\right) + z^2 = 1 - z(c - z),$$

where $$c = 2\cos\theta.$$

Take the logarithms of both members; thus

$$\log(1 - zx) + \log\left(1 - \frac{z}{x}\right) = \log\{1 - z(c - z)\};$$

therefore $$zx + \tfrac{1}{2}z^2x^2 + \tfrac{1}{3}z^3x^3 + \ldots + \frac{z}{x} + \frac{1}{2}\frac{z^2}{x^2} + \frac{1}{3}\frac{z^3}{x^3} + \ldots$$

$$= z(c - z) + \tfrac{1}{2}z^2(c - z)^2 + \tfrac{1}{3}z^3(c - z)^3 + \ldots + \frac{1}{n}z^n(c - z)^n + \ldots$$

In this identity we may equate the coefficients of z^n. On the left-hand side the coefficient of z^n is $\frac{1}{n}\left(x^n + \frac{1}{x^n}\right)$; that is, $\frac{2}{n}\cos n\theta$; the coefficient of z^n on the right-hand side must be obtained by picking out the coefficient of z^n from the expansion of $\frac{1}{n}z^n(c - z)^n$ and of the terms which precede it.

The coefficient of z^n in $\frac{1}{n}z^n(c - z)^n$ is $\frac{c^n}{n}$;

the coefficient of z^n in $\frac{z^{n-1}(c - z)^{n-1}}{n - 1}$ is $-\frac{1}{n - 1}(n - 1)c^{n-2}$;

the coefficient of z^n in $\frac{z^{n-2}(c - z)^{n-2}}{n - 2}$ is $\frac{1}{n - 2}\frac{(n - 2)(n - 3)}{1 \cdot 2}c^{n-4}$;

and generally the coefficient of z^n in $\frac{1}{n - r}z^{n-r}(c - z)^{n-r}$ is

$$\frac{(-1)^r}{n - r}\frac{(n - r)(n - r - 1)\ldots(n - 2r + 1)}{\lfloor r}c^{n-2r}.$$

Thus $2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1\,.\,2}(2\cos\theta)^{n-4} -$

$$\ldots + (-1)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{\underline{|r}}(2\cos\theta)^{n-2r} + \ldots$$

The series on the right hand is to continue so long as the powers of $2\cos\theta$ are not negative.

287. It is obvious either from the above series or from that in Art. 269, that when n is an *even positive* integer $\cos n\theta$ can be arranged in a series of powers of $\sin^2\theta$. Thus we may assume in this case

$$\cos n\theta = 1 + A_2\sin^2\theta + A_4\sin^4\theta + A_6\sin^6\theta + \ldots + A_n\sin^n\theta.$$

It is clear that the first term must be unity, because when $\theta = 0$ we have $\sin\theta = 0$ and $\cos n\theta = 1$. Now we shall adopt an indirect method of determining the values of the coefficients $A_2, A_4, \ldots\ldots$ Change θ into $\theta + h$; thus $\cos n\theta$ becomes

$$\cos n\theta\cos nh - \sin n\theta\sin nh;$$

now put for $\cos nh$ and $\sin nh$ their values in terms of nh by Art. 274; thus the above expression becomes

$$\cos n\theta - nh\sin n\theta - \frac{n^2h^2}{2}\cos n\theta + \&c.$$

Again in the term $A_{2r}\sin^{2r}\theta$ change θ into $\theta + h$; we thus get $A_{2r}(\sin\theta\cos h + \cos\theta\sin h)^{2r}$, that is,

$$A_{2r}\left(\sin\theta + h\cos\theta - \frac{h^2}{2}\sin\theta - \ldots\right)^{2r}.$$

If this be expanded in powers of h the term involving h^2 is

$$A_{2r}\left\{\frac{2r(2r-1)}{1\,.\,2}\sin^{2r-2}\theta\cos^2\theta - r\sin^{2r}\theta\right\}h^2.$$

Equate the coefficients of h^2; thus

$$-\frac{n^2}{2}\cos n\theta = A_2\{\cos^2\theta - \sin^2\theta\} + A_4\{2\,.\,3\sin^2\theta\cos^2\theta - 2\sin^4\theta\}$$

$$+ \ldots + A_{2r}\left\{\frac{2r(2r-1)}{1\,.\,2}\sin^{2r-2}\theta\cos^2\theta - r\sin^{2r}\theta\right\} + \ldots$$

Now put $1-\sin^2\theta$ for $\cos^2\theta$ on the right-hand side; then the term containing $\sin^{2r}\theta$ will be

$$-A_{2r}\left\{\frac{2r(2r-1)}{1\,.\,2}+r\right\}+A_{2r+2}\frac{(2r+2)(2r+1)}{1\,.\,2};$$

and this coefficient must be equal to that of $\sin^{2r}\theta$ in the series for $-\frac{n^2}{2}\cos n\theta$, that is, to $-\frac{n^2}{2}A_{2r}$; thus

$$\frac{n^2}{2}A_{2r}=2r^2A_{2r}-A_{2r+2}(r+1)(2r+1),$$

therefore $$A_{2r+2}=-\frac{n^2-(2r)^2}{(2r+1)(2r+2)}A_{2r}.$$

By means of this law we may form the coefficients in succession; we may consider $A_0=1$; then

$$A_2=-\frac{n^2}{1\,.\,2}A_0=-\frac{n^2}{1\,.\,2},$$

$$A_4=-\frac{n^2-2^2}{3\,.\,4}A_2=\frac{n^2(n^2-2^2)}{1\,.\,2\,.\,3\,.\,4},$$

and so on.

Hence, finally,

$$\cos n\theta=1-\frac{n^2}{1\,.\,2}\sin^2\theta+\frac{n^2(n^2-2^2)}{\lfloor 4}\sin^4\theta-\frac{n^2(n^2-2^2)(n^2-4^2)}{\lfloor 6}\sin^6\theta+\ldots$$

In the above process by equating the coefficients of h we shall obtain

$$-n\sin n\theta=A_2 2\sin\theta\cos\theta+A_4 4\sin^3\theta\cos\theta+\ldots+A_{2r}2r\sin^{2r-1}\theta\cos\theta+\ldots$$

Substitute the values of A_2, $A_4\ldots$; thus

$$\sin n\theta=n\cos\theta\left\{\sin\theta-\frac{n^2-2^2}{\lfloor 3}\sin^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{\lfloor 5}\sin^5\theta-\ldots\right\}$$

When n is odd, we may start by assuming

$$\sin n\theta=A_1\sin\theta+A_3\sin^3\theta+A_5\sin^5\theta+\ldots+A_n\sin^n\theta;$$

then, by proceeding as before, we shall find

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1)}{\underline{|3}}\sin^3\theta + \frac{n(n^2-1)(n^2-3^2)}{\underline{|5}}\sin^5\theta - \dots$$

$$\cos n\theta = \cos\theta\left\{1 - \frac{n^2-1}{1\,.\,2}\sin^2\theta + \frac{(n^2-1)(n^2-3^2)}{\underline{|4}}\sin^4\theta - \dots\right\}$$

288. In the four formulæ obtained in the preceding article change θ into $\frac{\pi}{2}-\theta$; thus we have, if n be an even integer,

$$(-1)^{\frac{n}{2}}\cos n\theta = 1 - \frac{n^2}{2}\cos^2\theta + \frac{n^2(n^2-2^2)}{\underline{|4}}\cos^4\theta - \dots$$

$$(-1)^{\frac{n}{2}+1}\sin n\theta = n\sin\theta\left\{\cos\theta - \frac{n^2-2^2}{\underline{|3}}\cos^3\theta + \frac{(n^2-2^2)(n^2-4^2)}{\underline{|5}}\cos^5\theta - \dots\right\};$$

and if n be an odd integer,

$$(-1)^{\frac{n-1}{2}}\cos n\theta = n\cos\theta - \frac{n(n^2-1)}{\underline{|3}}\cos^3\theta + \frac{n(n^2-1)(n^2-3^2)}{\underline{|5}}\cos^5\theta - \dots$$

$$(-1)^{\frac{n-1}{2}}\sin n\theta = \sin\theta\left\{1 - \frac{n^2-1}{1\,.\,2}\cos^2\theta + \frac{(n^2-1)(n^2-3^2)}{\underline{|4}}\cos^4\theta - \dots\right\}.$$

MISCELLANEOUS EXAMPLES.

1. Expand $(\sin\theta)^{4n+2}$ in terms of cosines of multiples of θ.

2. Expand $(\sin\theta)^{4n+1}$ in terms of sines of multiples of θ.

3. Expand $(\cos\theta)^{2n}$ in terms of cosines of multiples of θ.

4. Prove that in any triangle

$$\frac{a^2\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)} + \frac{b^2\cos\frac{1}{2}(C-A)}{\cos\frac{1}{2}(C+A)} + \frac{c^2\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = 2(ab+bc+ca).$$

5. From the angles of a triangle ABC, perpendiculars AD, BE, CF are let fall upon the opposite sides; prove that

$$a\sin(BAD-CAD) + b\sin(CBE-ABE) + c\sin(ACF-BCF) = 0.$$

6. From A, B draw AD, BD perpendiculars respectively to AC, BC. If ρ be the radius of the circle inscribed in ABD, then

$$AB = \rho\,(\sec A + \sec B + \tan A + \tan B).$$

7. Three equal circles of radius a touch each other; shew that the area of the space between them is

$$\left(\sqrt{3} - \frac{\pi}{2}\right) a^2.$$

8. The area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and of a circumscribed regular polygon of half the number of sides.

9. The area of a regular polygon circumscribed about a circle is an harmonic mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.

10. If the side of a pentagon inscribed in a circle be c, the radius is $\dfrac{c\sqrt{(5+\sqrt{5})}}{\sqrt{10}}$.

11. Three circles whose radii are a, b, c touch each other externally; prove that the tangents at the points of contact meet in a point whose distance from any one of them is

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

12. The sides taken in order of a quadrilateral whose opposite angles are supplementary are 3, 3, 4, 4; find the area and the radii of the inscribed and circumscribed circles.

13. The area of a regular polygon inscribed in a circle is to that of the circumscribed polygon of the same number of sides as 3 to 4; find the number of sides.

14. If the diameters of three circles which touch each other be a, b, c, and α, β, γ be the chords of the arcs between the points of contact in each, shew that

$$\frac{1}{\alpha\beta\gamma} = \left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{c} + \frac{1}{a}\right).$$

15. Shew that the limit of $\left(\frac{\tan\theta}{\theta}\right)^{\frac{1}{\theta^2}}$, when θ is indefinitely diminished, is e.

16. The two diagonals of a quadrilateral figure whose opposite angles are supplementary cannot be equal unless some one of the sides be equal to the opposite one.

17. Two circles whose radii are a and b cut one another at an angle γ; shew that the length of the common chord is

$$\frac{2ab\sin\gamma}{\sqrt{(a^2+2ab\cos\gamma+b^2)}}.$$

18. The radius of the circle inscribed in a triangle can never be greater than half the radius of the circle described about the triangle.

XXI. EXPONENTIAL VALUES OF THE COSINE AND SINE.

289. If we expand e^{kx} and e^{-kx} by the exponential theorem we obtain

$$\frac{1}{2}(e^{kx}+e^{-kx}) = 1 + \frac{k^2x^2}{1.2} + \frac{k^4x^4}{\lfloor 4} + \frac{k^6x^6}{\lfloor 6} + \dots$$

$$\frac{1}{2k}(e^{kx}-e^{-kx}) = x + \frac{k^2x^3}{\lfloor 3} + \frac{k^4x^5}{\lfloor 5} + \frac{k^6x^7}{\lfloor 7} + \dots$$

If it were possible to make $k^2=-1$, so that $k^4=1$, $k^6=-1$, and so on, then the right-hand member of the first equation would be the expansion of $\cos x$, and the right-hand member of the second equation would be the expansion of $\sin x$ (see Art. 274). Hence we are led to these results,

$$\cos x = \frac{e^{x\sqrt{(-1)}}+e^{-x\sqrt{(-1)}}}{2}, \quad \sin x = \frac{e^{x\sqrt{(-1)}}-e^{-x\sqrt{(-1)}}}{2\sqrt{(-1)}}.$$

The meaning of these equations is simply this; if we expand $e^{\sqrt{(-1)}x}$ and $e^{-\sqrt{(-1)}x}$, by the exponential theorem, in the same way as if $\sqrt{(-1)}$ were a real quantity, we shall by the above formulæ obtain the known series for $\cos x$ and $\sin x$.

These expressions for $\cos x$ and $\sin x$ are called the *exponential values* of the cosine and sine.

290. From the exponential values of the cosine and sine we may deduce similar values for the other trigonometrical functions. Thus, for example,

$$\tan x = \frac{e^{x\sqrt{(-1)}} - e^{-x\sqrt{(-1)}}}{\sqrt{(-1)}\,\{e^{x\sqrt{(-1)}} + e^{-x\sqrt{(-1)}}\}}.$$

We shall now use the exponential values in establishing certain results.

291. *To expand θ in powers of* $\tan\theta$.

By Art. 290, $$\sqrt{(-1)}\tan\theta = \frac{e^{\theta\sqrt{(-1)}} - e^{-\theta\sqrt{(-1)}}}{e^{\theta\sqrt{(-1)}} + e^{-\theta\sqrt{(-1)}}};$$

therefore $$\frac{1+\sqrt{(-1)}\tan\theta}{1-\sqrt{(-1)}\tan\theta} = \frac{e^{\theta\sqrt{(-1)}}}{e^{-\theta\sqrt{(-1)}}} = e^{2\theta\sqrt{(-1)}}.$$

Take the logarithms of both members; thus

$$2\theta\sqrt{(-1)} = \log\{1+\sqrt{(-1)}\tan\theta\} - \log\{1-\sqrt{(-1)}\tan\theta\}$$

$$= 2\sqrt{(-1)}\left\{\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots\right\};$$

therefore $$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$$

This is called *Gregory's Series.*

Let $\tan\theta = x$, so that $\theta = \tan^{-1}x$;

thus $$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

292. The preceding investigation is unsatisfactory, because it gives no indication of the extent to which the result may be relied upon as arithmetically intelligible and true. The n^{th} term of the last series is $\frac{(-1)^{n-1}x^{2n-1}}{2n-1}$; hence the numerical value of the ratio of the $(n+1)^{th}$ term to the n^{th} is $\frac{2n-1}{2n+1}x^2$; therefore the series is convergent if x be less than unity (*Algebra*, Art. 559). The series is also convergent when x is equal to unity (*Algebra*, Art. 558). For values of x greater than unity the series is not convergent, and is therefore not arithmetically intelligible.

293. Moreover $\tan^{-1}x$ has an infinite number of values corresponding to the same value of x, so that one member of what appears as an equation admits of more values than the other; this point is left unexplained in the investigation which has been given.

The subject of series cannot be adequately treated without using the Differential Calculus. The student must therefore be referred to treatises on that subject for a satisfactory demonstration of Gregory's Series. It is there shewn that so long as θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, the result $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$ is absolutely true. (See *Differential Calculus*, Chapter VII.)

If, however, $\theta = n\pi + \phi$, where ϕ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, then

$$\phi = \tan\phi - \frac{1}{3}\tan^3\phi + \frac{1}{5}\tan^5\phi - \dots;$$

that is, $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots$

294. In Gregory's Series put $\theta = \frac{\pi}{4}$; then since $\tan\frac{\pi}{4} = 1$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$$

This series might be used for calculating the value of π; but it is very slowly convergent, so that a large number of terms would have to be taken to calculate π to a close approximation.

295. *Euler's Series.*

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1=\frac{\pi}{4};$$

$$\text{thus } \frac{\pi}{4}=\frac{1}{2}-\frac{1}{3\,.\,2^3}+\frac{1}{5\,.\,2^5}-\frac{1}{7\,.\,2^7}+\dots$$

$$+\frac{1}{3}-\frac{1}{3\,.\,3^3}+\frac{1}{5\,.\,3^5}-\frac{1}{7\,.\,3^7}+\dots$$

296. *Machin's Series.* We shall first shew that

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}.$$

$$2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}=\tan^{-1}\frac{10}{24}=\tan^{-1}\frac{5}{12},$$

$$4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{5}{12}=\tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}}=\tan^{-1}\frac{120}{119}.$$

Hence $4\tan^{-1}\frac{1}{5}$ is a little greater than $\frac{\pi}{4}$; suppose

$$4\tan^{-1}\frac{1}{5}=\frac{\pi}{4}+\tan^{-1}x,$$

then
$$\frac{120}{119}=\tan\left(\frac{\pi}{4}+\tan^{-1}x\right)=\frac{1+x}{1-x};$$

from this we find $x=\frac{1}{239}$.

Therefore $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= 4\left\{\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots\right\}$$

$$-\left\{\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \dots\right\}.$$

297. It may be shewn that

$$\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99};$$

thus $$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}.$$

The series for $\tan^{-1}\frac{1}{70}$ and $\tan^{-1}\frac{1}{99}$ are convenient, for purposes of numerical calculation.

The value of π has been calculated by two computers independently to 440 places of decimals (see *Lady's and Gentleman's Diary* for 1854, page 70, and for 1855, page 86).

298. *Given* $\sin x = n \sin (x + a)$, *required to expand* x *in powers of* n.

Here $$e^{x\sqrt{(-1)}} - e^{-x\sqrt{(-1)}} = n\{e^{(x+a)\sqrt{(-1)}} - e^{-(x+a)\sqrt{(-1)}}\},$$

therefore $$e^{2x\sqrt{(-1)}} - 1 = n\{e^{(2x+a)\sqrt{(-1)}} - e^{-a\sqrt{(-1)}}\},$$

therefore $$e^{2x\sqrt{(-1)}}\{1 - ne^{a\sqrt{(-1)}}\} = 1 - ne^{-a\sqrt{(-1)}},$$

therefore $$e^{2x\sqrt{(-1)}} = \frac{1 - ne^{-a\sqrt{(-1)}}}{1 - ne^{a\sqrt{(-1)}}},$$

therefore $$2x\sqrt{(-1)} = \log\{1 - ne^{-a\sqrt{(-1)}}\} - \log\{1 - ne^{a\sqrt{(-1)}}\}$$

$$= n\{e^{a\sqrt{(-1)}} - e^{-a\sqrt{(-1)}}\} + \frac{n^2}{2}\{e^{2a\sqrt{(-1)}} - e^{-2a\sqrt{(-1)}}\} + \frac{n^3}{3}\{e^{3a\sqrt{(-1)}} - e^{-3a\sqrt{(-1)}}\} + \dots$$

therefore $$x = n\sin a + \frac{n^2}{2}\sin 2a + \frac{n^3}{3}\sin 3a + \dots\dots$$

As an example, suppose $a = \pi - 2x$, then $n = 1$; thus

$$x = \sin 2x - \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x - \frac{1}{4}\sin 8x + \ldots$$

299. *Given* $\tan x = n \tan y$, *required to find a series for* x.

Here $$\frac{e^{x\sqrt{(-1)}} - e^{-x\sqrt{(-1)}}}{e^{x\sqrt{(-1)}} + e^{-x\sqrt{(-1)}}} = n\frac{e^{y\sqrt{(-1)}} - e^{-y\sqrt{(-1)}}}{e^{y\sqrt{(-1)}} + e^{-y\sqrt{(-1)}}};$$

therefore $$\frac{e^{2x\sqrt{(-1)}} - 1}{e^{2x\sqrt{(-1)}} + 1} = n\frac{e^{2y\sqrt{(-1)}} - 1}{e^{2y\sqrt{(-1)}} + 1};$$

therefore $$e^{2x\sqrt{(-1)}} = \frac{(1+n)e^{2y\sqrt{(-1)}} + 1 - n}{(1-n)e^{2y\sqrt{(-1)}} + 1 + n}$$

$$= e^{2y\sqrt{(-1)}} \times \frac{1 + me^{-2y\sqrt{(-1)}}}{1 + me^{2y\sqrt{(-1)}}}, \text{ where } m = \frac{1-n}{1+n};$$

therefore

$$2x\sqrt{(-1)} = 2y\sqrt{(-1)} + \log\{1 + me^{-2y\sqrt{(-1)}}\} - \log\{1 + me^{2y\sqrt{(-1)}}\}$$

$$= 2y\sqrt{(-1)} - m\{e^{2y\sqrt{(-1)}} - e^{-2y\sqrt{(-1)}}\} + \frac{m^2}{2}\{e^{4y\sqrt{(-1)}} - e^{-4y\sqrt{(-1)}}\} - \ldots;$$

therefore $$x = y - m\sin 2y + \frac{m^2}{2}\sin 4y - \frac{m^3}{3}\sin 6y + \ldots$$

300. *To find the coefficient of* x^n *in the expansion of* $e^{ax}\cos bx$ *in powers of* x.

Here $e^{ax}\cos bx = \frac{1}{2}e^{ax}\{e^{bx\sqrt{(-1)}} + e^{-bx\sqrt{(-1)}}\} = \frac{1}{2}e^{\{a+b\sqrt{(-1)}\}x} + \frac{1}{2}e^{\{a-b\sqrt{(-1)}\}x}$.

Expand these two exponential expressions by the exponential theorem; then the coefficient of x^n is

$$\frac{1}{2\lfloor n}[\{a + b\sqrt{(-1)}\}^n + \{a - b\sqrt{(-1)}\}^n]$$

$$= \frac{r^n}{2\lfloor n}\left[\left\{\frac{a}{r} + \frac{b}{r}\sqrt{(-1)}\right\}^n + \left\{\frac{a}{r} - \frac{b}{r}\sqrt{(-1)}\right\}^n\right].$$

Now suppose $\frac{a}{r} = \cos\theta$, $\frac{b}{r} = \sin\theta$, so that $r^2 = a^2 + b^2$.

Thus the coefficient of x^n becomes

$$\frac{(a^2+b^2)^{\frac{n}{2}}}{2\lfloor n}[\{\cos\theta+\sqrt{(-1)}\sin\theta\}^n+\{\cos\theta-\sqrt{(-1)}\sin\theta\}^n]$$

$$=\frac{(a^2+b^2)^{\frac{n}{2}}}{2\lfloor n}[\cos n\theta+\sqrt{(-1)}\sin n\theta+\cos n\theta-\sqrt{(-1)}\sin n\theta]$$

$$=\frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\cos n\theta.$$

301. The series in Art. 298 may sometimes be of assistance in the solution of triangles.

We have $\sin B=\frac{b}{a}\sin A=\frac{b}{a}\sin(B+C)$;

henee, by the formula,

$$B=\frac{b}{a}\sin C+\frac{b^2}{2a^2}\sin 2C+\frac{b^3}{3a^3}\sin 3C+\dots$$

If b be less than a the series is convergent, and if $\frac{b}{a}$ be a small fraction a few terms of this series may give B to a sufficient degree of approximation; the series gives the *circular measure* of B, and the measure in degrees or minutes or seconds may be deduced by Art. 22.

302. *Given two sides of a triangle and the included angle, to find a series for the logarithm of the third side.*

Suppose a and b the given sides and C the circular measure of the given angle; suppose b less than a, we have

$$c^2=a^2+b^2-2ab\cos C=a^2+b^2-ab\{e^{C\sqrt{(-1)}}+e^{-C\sqrt{(-1)}}\}$$

$$=\{a-be^{C\sqrt{(-1)}}\}\{a-be^{-C\sqrt{(-1)}}\}$$

$$=a^2\left\{1-\frac{b}{a}e^{C\sqrt{(-1)}}\right\}\left\{1-\frac{b}{a}e^{-C\sqrt{(-1)}}\right\}$$

thus $2\log c = 2\log a + \log\left\{1 - \frac{b}{a}e^{C\sqrt{(-1)}}\right\} + \log\left\{1 - \frac{b}{a}e^{-C\sqrt{(-1)}}\right\}$

$$= 2\log a - \frac{b}{a}\left\{e^{C\sqrt{(-1)}} + e^{-C\sqrt{(-1)}}\right\} - \frac{b^2}{2a^2}\left\{e^{2C\sqrt{(-1)}} + e^{-2C\sqrt{(-1)}}\right\} - \dots;$$

therefore $\log c = \log a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \dots$

This series is convergent since b is supposed less than a, and if $\frac{b}{a}$ be small a few terms may give $\log c$ to a sufficient degree of approximation.

EXAMPLES.

1. Apply the exponential values of the sine and cosine to shew that

$$\frac{\sin A}{1 - \cos A} = \cot\frac{A}{2}.$$

2. If the sides of a right-angled triangle be 49 and 51, shew that the angles opposite to them are 43° 51′ 15″ and 46° 8′ 45″ nearly.

3. If the angle C of a triangle be given, and the other two adjacent sides a, b be nearly equal, shew that the other angles are nearly equal to

$$90^\circ - \frac{C}{2} \pm \frac{180^\circ}{\pi}\left\{\frac{a-b}{a+b}\cot\frac{C}{2} - \frac{1}{3}\left(\frac{a-b}{a+b}\cot\frac{C}{2}\right)^3\right\}.$$

4. In any triangle, if $A - B$ be small compared with C,

$$A = B + 2\frac{a-b}{c}\sin B + \left(\frac{a-b}{c}\right)^2 \sin 2B \text{ nearly.}$$

5. If a and b be the sides of a plane triangle, A and B the opposite angles, then will $\log b - \log a$

$$= \cos 2A - \cos 2B + \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}(\cos 6A - \cos 6B) + \dots$$

6. Shew that $\frac{\pi}{8} = \frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \ldots$

7. If $A + B\sqrt{(-1)} = \log\{m + n\sqrt{(-1)}\}$, shew that

$$\tan B = \frac{n}{m}, \text{ and } 2A = \log(n^2 + m^2).$$

8. Reduce $\cos\{\theta + \phi\sqrt{(-1)}\}$ to the form $\alpha + \beta\sqrt{(-1)}$.

9. Reduce $\sin\{\theta + \phi\sqrt{(-1)}\}$ to the form $\alpha + \beta\sqrt{(-1)}$.

10. Reduce $\{a + b\sqrt{(-1)}\}^{\frac{1}{p+q\sqrt{(-)}}}$ to the form $\alpha + \beta\sqrt{(-1)}$.

11. Reduce $\{a + b\sqrt{(-1)} + c^{\theta\sqrt{(-1)}}\}^{p+q\sqrt{(-1)}}$ to the form $\alpha + \beta\sqrt{(-1)}$.

12. Prove that

$$\{\sin(\alpha - \theta) + e^{\pm\alpha\sqrt{(-1)}}\sin\theta\}^n = \sin^{n-1}\alpha\,\{\sin(\alpha - n\theta) + e^{\pm\alpha\sqrt{(-1)}}\sin n\theta\}.$$

XXII. SUMMATION OF TRIGONOMETRICAL SERIES.

303. *To find the sum of the sines of a series of angles which are in arithmetical progression.*

Let the proposed series consist of the following n terms,

$$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin\{\alpha + (n-1)\beta\}.$$

We have

$$\cos\left(\alpha - \frac{1}{2}\beta\right) - \cos\left(\alpha + \frac{1}{2}\beta\right) = 2\sin\frac{1}{2}\beta\sin\alpha,$$

$$\cos\left(\alpha + \frac{1}{2}\beta\right) - \cos\left(\alpha + \frac{3}{2}\beta\right) = 2\sin\frac{1}{2}\beta\sin(\alpha + \beta),$$

$$\cos\left(\alpha + \frac{3}{2}\beta\right) - \cos\left(\alpha + \frac{5}{2}\beta\right) = 2\sin\frac{1}{2}\beta\sin(\alpha + 2\beta),$$

....

$$\cos\left(\alpha+\frac{2n-3}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2\sin\frac{1}{2}\beta\sin\{\alpha+(n-1)\beta\}.$$

Let S denote the sum of the proposed series; then, by addition,

$$\cos\left(\alpha-\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2S\sin\frac{1}{2}\beta;$$

therefore $$S=\frac{\cos\left(\alpha-\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)}{2\sin\frac{1}{2}\beta}$$

$$=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}.$$

304. *To find the sum of the cosines of a series of angles which are in arithmetical progression.*

Let the proposed series consist of the following n terms,

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}.$$

We have

$$\sin\left(\alpha+\frac{1}{2}\beta\right)-\sin\left(\alpha-\frac{1}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos\alpha,$$

$$\sin\left(\alpha+\frac{3}{2}\beta\right)-\sin\left(\alpha+\frac{1}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos(\alpha+\beta),$$

$$\sin\left(\alpha+\frac{5}{2}\beta\right)-\sin\left(\alpha+\frac{3}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos(\alpha+2\beta),$$

....

$$\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos\{\alpha+(n-1)\beta\}.$$

Let S denote the proposed series; then, by addition,

$$\sin\left(\alpha + \frac{2n-1}{2}\beta\right) - \sin\left(\alpha - \frac{1}{2}\beta\right) = 2S\sin\frac{1}{2}\beta;$$

therefore
$$S = \frac{\sin\left(\alpha + \frac{2n-1}{2}\beta\right) - \sin\left(\alpha - \frac{1}{2}\beta\right)}{2\sin\frac{1}{2}\beta}$$

$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}.$$

305. The series in Art. 304 may be deduced from that in Art. 303 by writing $\alpha + \frac{\pi}{2}$ for α; the sums of these series are required so often in the solution of problems, that the student should be able to quote them from memory. As we have just intimated, if the first result be known it is sufficient, since the second can be obtained from the first by changing *sine* into *cosine* in the first factor of the numerator. It will be seen that the results are obviously correct when $n=1$, and when $n=2$; thus there is a test of the accuracy with which the formulæ are quoted. The cases in which $\beta = \alpha$ may be specially noticed; we have then

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha = \frac{\sin\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}},$$

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha = \frac{\cos\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

306. We may now deduce the sum of the following n terms:
$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots + (-1)^{n-1}\sin\{\alpha + (n-1)\beta\}$.

This series may be written

$$\sin\alpha+\sin(\alpha+\beta+\pi)+\sin(\alpha+2\beta+2\pi)+\ldots+\sin\{\alpha+(n-1)(\beta+\pi)\}.$$

We have then only to change β into $\beta+\pi$ in the result of Art. 303.

Hence the required sum is $\dfrac{\sin\left\{\alpha+\dfrac{(n-1)(\beta+\pi)}{2}\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\sin\dfrac{\beta+\pi}{2}}$.

Similarly

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\ldots+(-1)^{n-1}\cos\{\alpha+(n-1)\beta\}$$

$$=\frac{\cos\left\{\alpha+\dfrac{(n-1)(\beta+\pi)}{2}\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\sin\dfrac{\beta+\pi}{2}}.$$

307. *To find the sum of the following* n *terms.*

$$\operatorname{cosec} x+\operatorname{cosec} 2x+\operatorname{cosec} 4x+\operatorname{cosec} 8x+\ldots+\operatorname{cosec} 2^{n-1}x.$$

We have $\operatorname{cosec} x=\cot\dfrac{x}{2}-\cot x,$

$$\operatorname{cosec} 2x=\cot x-\cot 2x,$$

....

$$\operatorname{cosec} 2^{n-1}x=\cot 2^{n-2}x-\cot 2^{n-1}x.$$

Let S denote the proposed series; then, by addition,

$$S=\cot\frac{x}{2}-\cot 2^{n-1}x.$$

308. *To find the sum of the following* n *terms.*

$$\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots+\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}.$$

We have $\tan x = \cot x - 2\cot 2x$,

$$\frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cot\frac{x}{2} - \cot x,$$

$$\frac{1}{2^2}\tan\frac{x}{2^2} = \frac{1}{2^2}\cot\frac{x}{2^2} - \frac{1}{2}\cot\frac{x}{2},$$

....

$$\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{x}{2^{n-2}}.$$

Let S denote the proposed series; then, by addition,

$$S = \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} - 2\cot 2x.$$

The term $\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} = \frac{1}{x}\cos\beta\,\frac{\beta}{\sin\beta}$, where $\beta = \frac{x}{2^{n-1}}$; if we suppose n to increase indefinitely, $\cos\beta = 1$, and $\frac{\beta}{\sin\beta} = 1$.

Thus the limit of the proposed series, when n is indefinitely increased, is $\frac{1}{x} - 2\cot 2x$.

309. *To find the sum of the following* n *terms.*

$$\sin\alpha + c\sin(\alpha+\beta) + c^2\sin(\alpha+2\beta) + \ldots + c^{n-1}\sin\{\alpha+(n-1)\beta\}.$$

Let S denote the proposed series; substitute for the sines their exponential values, and let k stand for $\sqrt{(-1)}$; thus

$$2kS = e^{\alpha k} + ce^{(\alpha+\beta)k} + c^2e^{(\alpha+2\beta)k} + \ldots + c^{n-1}e^{(\alpha+n\beta-\beta)k}$$
$$- e^{-\alpha k} - ce^{-(\alpha+\beta)k} - c^2e^{-(\alpha+2\beta)k} - \ldots - c^{n-1}e^{-(\alpha+n\beta-\beta)k}.$$

We have now two geometrical progressions; thus

$$2kS = e^{\alpha k}\frac{1-c^ne^{n\beta k}}{1-ce^{\beta k}} - e^{-\alpha k}\frac{1-c^ne^{-n\beta k}}{1-ce^{-\beta k}}$$

$$= \frac{e^{\alpha k}-e^{-\alpha k}-c\{e^{(\alpha-\beta)k}-e^{-(\alpha-\beta)k}\}-c^n\{e^{(\alpha+n\beta)k}-e^{-(\alpha+n\beta)k}\}+c^{n+1}\{e^{(n\beta+\alpha-\beta)k}-e^{-(n\beta+\alpha-\beta)k}\}}{1-c(e^{\beta k}+e^{-\beta k})+c^2};$$

therefore

$$S = \frac{\sin a - c \sin (a - \beta) - c^n \sin (a + n\beta) + c^{n+1} \sin \{a + (n-1)\beta\}}{1 - 2c \cos \beta + c^2}.$$

If c be less than unity, then when n is indefinitely increased c^n and c^{n+1} diminish without limit; hence if c be less than unity, the limit of the proposed series when n is indefinitely increased is

$$\frac{\sin a - c \sin (a - \beta)}{1 - 2c \cos \beta + c^2}.$$

Similarly we can shew that

$$\cos a + c \cos (a + \beta) + c^2 \cos (a + 2\beta) + \ldots + c^{n-1} \cos \{a + (n-1)\beta\}$$

$$= \frac{\cos a - c \cos (a - \beta) - c^n \cos (a + n\beta) + c^{n+1} \cos \{a + (n-1)\beta\}}{1 - 2c \cos \beta + c^2}.$$

This result may also be obtained from the preceding by changing a into $a + \frac{\pi}{2}$. If c be less than unity the limit of the proposed series, when n is indefinitely increased, is

$$\frac{\cos a - c \cos (a - \beta)}{1 - 2c \cos \beta + c^2}.$$

310. To sum the infinite series

$$c \sin (a + \beta) + \frac{c^2}{1 \cdot 2} \sin (a + 2\beta) + \frac{c^3}{\underline{|3}} \sin (a + 3\beta) + \ldots$$

Let S denote the proposed series; substitute for the sines their exponential values, and let k stand for $\sqrt{(-1)}$; thus

$$2kS = c e^{(a+\beta)k} + \frac{c^2}{1 \cdot 2} e^{(a+2\beta)k} + \frac{c^3}{\underline{|3}} e^{(a+3\beta)k} + \ldots$$

$$- c e^{-(a+\beta)k} - \frac{c^2}{1 \cdot 2} e^{-(a+2\beta)k} - \frac{c^3}{\underline{|3}} e^{-(a+3\beta)k} - \ldots$$

$$= e^{ak} \{e^{ce^{\beta k}} - 1\} - e^{-ak} \{e^{ce^{-\beta k}} - 1\}.$$

Now $e^{\beta k} = \cos\beta + k\sin\beta$, $e^{-\beta k} = \cos\beta - k\sin\beta$;

thus $2kS = e^{c\cos\beta + k(\alpha + c\sin\beta)} - e^{c\cos\beta - k(\alpha + c\sin\beta)} - (e^{\alpha k} - e^{-\alpha k})$

$$= e^{c\cos\beta}\left\{e^{k(\alpha + c\sin\beta)} - e^{-k(\alpha + c\sin\beta)}\right\} - 2k\sin\alpha;$$

therefore $S = e^{c\cos\beta}\sin(\alpha + c\sin\beta) - \sin\alpha$.

Similarly it may be shewn that the sum of the infinite series

$$c\cos(\alpha + \beta) + \frac{c^2}{1\,.\,2}\cos(\alpha + 2\beta) + \frac{c^3}{\underline{|3}}\cos(\alpha + 3\beta) + \ldots$$

is
$$e^{c\cos\beta}\cos(\alpha + c\sin\beta) - \cos\alpha.$$

This result may also be obtained from the preceding by changing α into $\alpha + \frac{\pi}{2}$.

311. We shall not solve any more examples of the summation of Trigonometrical Series; the student will find more exercise of this kind in the collection of examples for practice. In many cases the summation is effected by the artifice which is employed in Arts. 307, 308, by which each term of the proposed series is resolved into the *difference* of two terms. Practice alone will give the student readiness in effecting such transformations. If he cannot discover the necessary mode of resolution in any example, he will find no difficulty in recognising it when he sees the *result* of the summation given in the collection of answers. Thus, for example, required the sum of the following n terms:

$$\sec\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \sec 3\alpha\sec 4\alpha + \ldots + \sec n\alpha\sec(n+1)\alpha.$$

The result is $\operatorname{cosec}\alpha\{\tan(n+1)\alpha - \tan\alpha\}$; and *by putting* $n = 1$ *this suggests the necessary transformation*, namely,

$$\sec\alpha\sec 2\alpha = \operatorname{cosec}\alpha\{\tan 2\alpha - \tan\alpha\};$$

then,
$$\sec 2\alpha\sec 3\alpha = \operatorname{cosec}\alpha\{\tan 3\alpha - \tan 2\alpha\},$$

and so on.

312. The student who is acquainted with the Differential and Integral Calculus, will be able to deduce numerous series from known series by differentiation or integration; and when the results are obtained they can frequently be established by more elementary methods. Thus, for example, differentiate both members of the equality established in Art. 308; then

$$\sec^2 x + \frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \dots + \frac{1}{2^{2n-2}}\sec^2\frac{x}{2^{n-1}}$$

$$= -\frac{1}{2^{2n-2}}\operatorname{cosec}^2 2^{n-1}x + 4\operatorname{cosec}^2 2x.$$

Again in Art. 309 put $\alpha = \beta$; thus

$$\frac{\sin\alpha}{1 - 2c\cos\alpha + c^2} = \sin\alpha + c\sin 2\alpha + c^2\sin 3\alpha + c^4\sin 4\alpha + \dots$$

Integrate with respect to α; thus

$$-\frac{1}{2c}\log(1 - 2c\cos\alpha + c^2) = \cos\alpha + \frac{c}{2}\cos 2\alpha + \frac{c^2}{3}\cos 3\alpha + \frac{c^3}{4}\cos 4\alpha + \dots$$

No constant is required; for when α is zero both sides are equal.

EXAMPLES.

1. Find the sum of n terms of the series

$$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \dots$$

2. Find the sum of n terms of the series

$$\sin^3\alpha + \sin^3(\alpha+\beta) + \sin^3(\alpha+2\beta) + \dots$$

3. Find the sum of n terms of the series

$$\cos^4\alpha + \cos^4(\alpha+\beta) + \cos^4(\alpha+2\beta) + \dots$$

4. Shew that

$$\tan n\theta = \frac{\sin\theta + \sin 3\theta + \sin 5\theta + \dots \text{ to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \dots \text{ to } n \text{ terms}}.$$

5. Sum to n terms the series

$$\cos\theta\cos(\theta+a)+\cos(\theta+a)\cos(\theta+2a)+\cos(\theta+2a)\cos(\theta+3a)+\dots$$

6. Shew that

$$\frac{\sin\theta-\sin 2\theta+\sin 3\theta-\dots \text{ to } n \text{ terms}}{\cos\theta-\cos 2\theta+\cos 3\theta-\dots \text{ to } n \text{ terms}}=\tan\frac{n+1}{2}(\pi+\theta).$$

7. Sum to n terms the series

$$\sin(n+1)\theta\cos\theta+\sin(n+2)\theta\cos 2\theta+\dots$$

8. Sum to n terms the series

$$\sin a\sin 2a+\sin 2a\sin 3a+\sin 3a\sin 4a+\dots$$

and thence deduce the sum to n terms of the series

$$1.2+2.3+3.4+\dots$$

9. Sum to n terms the series

$$\sin 3\theta\sin\theta+\sin 6\theta\sin 2\theta+\sin 12\theta\sin 4\theta+\dots$$

Sum to infinity the following series contained in the examples from 10 to 16 inclusive:

10. $\cos\theta+\frac{\cos\theta}{1}\cos 2\theta+\frac{\cos^2\theta}{1.2}\cos 3\theta+\frac{\cos^3\theta}{\lfloor 3}\cos 4\theta+\dots$

11. $\sin\theta-\frac{\sin 2\theta}{1.2}+\frac{\sin 3\theta}{\lfloor 3}-\dots$

12. $1-\frac{\cos 2\theta}{1.2}+\frac{\cos 4\theta}{\lfloor 4}-\dots$

13. $2\cos\theta+\frac{3}{2}\cos^2\theta+\frac{4}{3}\cos^3\theta+\frac{5}{4}\cos^4\theta+\dots$

14. $\sin\theta\cos\theta+\frac{\sin 2\theta\cos^2\theta}{1.2}+\frac{\sin 3\theta\cos^3\theta}{\lfloor 3}+\dots$

15. $\sin\theta+\frac{\sin\theta}{1}\cos 2\theta+\frac{\sin^2\theta}{1.2}\cos 3\theta+\dots$

Shew that $\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\dots=\log\left(2\cos\frac{\theta}{2}\right)$.

17. Shew that $\cos 2\theta + \frac{1}{3}\cos 6\theta + \frac{1}{5}\cos 10\theta + \ldots = \frac{1}{2}\log(\cot\theta)$.

18. Shew that

$$x\sin\theta - \frac{x^2\sin 2\theta}{2} + \frac{x^3\sin 3\theta}{3} - \ldots = \cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x} + \cot\theta\right).$$

19. Shew that

$$\log\cos\theta + \log\cos\frac{\theta}{2} + \log\cos\frac{\theta}{2^2} + \ldots = \log\left(\frac{\sin 2\theta}{2\theta}\right).$$

Sum the following series to n terms contained in the examples from 20 to 33 inclusive:

20. $\sin\theta\left(\sin\frac{\theta}{2}\right)^2 + 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2 + 4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2 + \ldots$

21. $\tan\frac{\theta}{2}\sec\theta + \tan\frac{\theta}{4}\sec\frac{\theta}{2} + \tan\frac{\theta}{8}\sec\frac{\theta}{4} + \ldots$

22. $\cot\theta\operatorname{cosec}\theta + 2\cot 2\theta\operatorname{cosec}2\theta + 2^2\cot 2^2\theta\operatorname{cosec}2^2\theta + \ldots$

23. $\frac{1}{\sin\theta\sin 2\theta} + \frac{1}{\sin 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\sin 4\theta} + \ldots$

24. $\frac{1}{\sin\theta\cos 2\theta} + \frac{1}{\cos 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\cos 4\theta} + \ldots$

25. $\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^2} + \ldots$

26. $\tan^{-1}x + \tan^{-1}\frac{x}{1+1.2.x^2} + \tan^{-1}\frac{x}{1+2.3.x^2} + \ldots$

27. $\sin\alpha\sin 3\alpha + \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} + \sin\frac{\alpha}{2^2}\sin\frac{3\alpha}{2^2} + \ldots$

28. $\frac{1}{\cos\theta+\cos 3\theta} + \frac{1}{\cos\theta+\cos 5\theta} + \frac{1}{\cos\theta+\cos 7\theta} + \ldots$

29. $\frac{\sin\theta}{\cos 2\theta+\cos\theta} + \frac{\sin 2\theta}{\cos 4\theta+\cos\theta} + \frac{\sin 3\theta}{\cos 6\theta+\cos\theta} + \ldots$

30. $\dfrac{\sin\theta}{1+2\cos\theta}+\dfrac{3\sin 3\theta}{1+2\cos 3\theta}+\dfrac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta}+\dots$

31. $\cot^{-1}(2a^{-1}+a)+\cot^{-1}(2a^{-1}+3a)+\cot^{-1}(2a^{-1}+6a)$
$+\cot^{-1}(2a^{-1}+10a)+\dots$

32. $\dfrac{1}{2}\sec\theta+\dfrac{1}{2^2}\sec\theta\sec 2\theta+\dfrac{1}{2^3}\sec\theta\sec 2\theta\sec 2^2\theta+\dots$

33. $\dfrac{1}{2}\log\tan 2\theta+\dfrac{1}{2^2}\log\tan 2^2\theta+\dfrac{1}{2^3}\log\tan 2^3\theta+\dots$

34. An equilateral polygon is inscribed in a circle and from any point in the circumference chords are drawn to the angular points; find the sum of the squares of the chords and the sum of the fourth powers of the chords.

35. Circles are inscribed in triangles, whose bases are the sides of a regular polygon of n sides, and whose vertices lie in one of the angular points; shew that the sum of the radii of the circles is

$$2r\left(1-n\sin^2\frac{\pi}{2n}\right),$$

where r is the radius of the circle circumscribing the polygon.

36. Circles are inscribed in triangles whose bases are the sides of a regular polygon of n sides and whose vertices lie in one of the angular points; shew that the sum of the areas of the circles is

$$16\pi r^2\sin^2\frac{\pi}{2n}\left\{\frac{n}{4}\sin^2\frac{\pi}{2n}+\frac{n-4}{8}\right\},$$

where r is the radius of the circle circumscribing the polygon.

37. Shew that if n be a positive integer

$$(n+1)n\sin\theta+n(n-1)\sin 2\theta+(n-1)(n-2)\sin 3\theta+\dots+2.1\sin n\theta$$
$$=\frac{n(n+3)}{2}\cot\frac{\theta}{2}-\frac{1}{4}\operatorname{cosec}^3\frac{\theta}{2}\left\{\cos\frac{3\theta}{2}-\cos\frac{2n+3}{2}\theta\right\}.$$

XXIII. RESOLUTION OF TRIGONOMETRICAL EXPRESSIONS INTO FACTORS.

313. It is known from treatises on the Theory of Equations that the expression $x^n - 1$, where n is a positive integer, can be resolved into n factors, each of the form $x - a$, where a is either a real quantity or an expression of the form $\alpha + \beta\sqrt{(-1)}$, where α and β are real; and there is only one such set of factors. We proceed now to resolve the expression $x^n - 1$, and some similar expressions, into component factors. The factors of the *expression* $x^n - 1$ are found by solving the *equation* $x^n - 1 = 0$; every root of the equation a determines one factor of the expression, namely $x - a$.

314. *To resolve* $x^n - 1$ *into factors.*

The expression $\cos\frac{2r\pi}{n} \pm \sqrt{(-1)}\sin\frac{2r\pi}{n}$, where r is any integer, is a root of the equation $x^n = 1$; for the n^{th} power of this expression is by De Moivre's Theorem $\cos 2r\pi \pm \sqrt{(-1)}\sin 2r\pi$, that is 1. First suppose n even. If we put $r = 0$ we obtain a real root 1, and the corresponding factor is $x - 1$; if we put $r = \frac{n}{2}$ we obtain a real root -1, and the corresponding factor is $x + 1$. If we put for r in succession the values 1, 2, 3, ... $\frac{n}{2} - 1$ we obtain $n - 2$ additional roots, since each value of r gives rise to two roots. These roots are all different, for the angles are less than π and all different, and thus $\cos\frac{2r\pi}{n}$ cannot have two coincident values.

Therefore $$x^n - 1 = (x - 1)(x + 1)\,P,$$

where P is the product of $n-2$ factors obtained by ascribing to r in succession the values 1, 2, 3, ... $\frac{n}{2}-1$ in the expression

$$x-\cos\frac{2r\pi}{n}\mp\sqrt{(-1)}\sin\frac{2r\pi}{n}.$$

The two factors

$$x-\cos\frac{2r\pi}{n}-\sqrt{(-1)}\sin\frac{2r\pi}{n}, \text{ and } x-\cos\frac{2r\pi}{n}+\sqrt{(-1)}\sin\frac{2r\pi}{n},$$

produce by multiplication the possible quadratic factor

$$\left(x-\cos\frac{2r\pi}{n}\right)^2+\sin^2\frac{2r\pi}{n}, \text{ that is, } x^2-2x\cos\frac{2r\pi}{n}+1.$$

Hence when n is even

$$x^n-1=(x-1)(x+1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\ldots$$
$$\ldots\left\{x^2-2x\cos\frac{(n-4)\pi}{n}+1\right\}\left\{x^2-2x\cos\frac{(n-2)\pi}{n}+1\right\}\ldots.(1).$$

Secondly, suppose n odd. The only real root of $x^n=1$ is now 1; the other $n-1$ roots are obtained by giving to r in succession the values 1, 2, 3, ... $\frac{n-1}{2}$ in the expression

$$\cos\frac{2r\pi}{n}\pm\sqrt{(-1)}\sin\frac{2r\pi}{n}.$$

Hence when n is odd

$$x^n-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\ldots$$
$$\ldots\left\{x^2-2x\cos\frac{(n-3)\pi}{n}+1\right\}\left\{x^2-2x\cos\frac{(n-1)\pi}{n}+1\right\}\ldots(2).$$

315. *To resolve* x^n+1 *into factors.*

The expression $\cos\frac{2r+1}{n}\pi\pm\sqrt{(-1)}\sin\frac{2r+1}{n}\pi$, where r is any integer, is a root of the equation $x^n=-1$; for the n^{th} power of this expression is $\cos(2r+1)\pi \pm \sqrt{(-1)}\sin(2r+1)\pi$, by De Moivre's Theorem, that is, -1. First, suppose n even; there is no real root of the equation $x^n = -1$; the n roots are all imaginary, and are found by giving to r in succession the values 0, 1, 2, 3, ... $\frac{n}{2}-1$, in the expression $\cos\frac{(2r+1)\pi}{n} \pm \sqrt{(-1)}\sin\frac{2r+1}{n}\pi$.

The two factors, $x - \cos\frac{2r+1}{n}\pi - \sqrt{(-1)}\sin\frac{2r+1}{n}\pi$,

and $$x - \cos\frac{2r+1}{n}\pi + \sqrt{(-1)}\sin\frac{2r+1}{n}\pi,$$

produce by multiplication the possible quadratic factor

$\left(x - \cos\frac{2r+1}{n}\pi\right)^2 + \sin^2\frac{2r+1}{n}\pi$, that is, $x^2 - 2x\cos\frac{2r+1}{n}\pi + 1$.

Hence when n is even

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{5\pi}{n} + 1\right)\dots$$
$$\dots\left(x^2 - 2x\cos\frac{n-3}{n}\pi + 1\right)\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right)\dots(1).$$

Secondly, suppose n odd. The only real root of $x^n = -1$ is -1; the other $n-1$ roots are obtained by giving to r in succession the values 0, 1, 2, 3, ... $\frac{n-3}{2}$ in the expression

$$\cos\frac{(2r+1)\pi}{n} \pm \sqrt{(-1)}\sin\frac{(2r+1)\pi}{n}.$$

Hence when n is odd

$$x^n + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\dots$$
$$\dots\left(x^2 - 2x\cos\frac{n-4}{n}\pi + 1\right)\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)\dots\dots(2).$$

316. The four formulæ established in the two preceding articles are *identically* true; we may deduce many particular

results by supposing particular values assigned to x. Thus in (1) of Art. 314, divide both sides by $x-1$; the quotient on the left-hand side will be $x^{n-1}+x^{n-2}+\ldots+x+1$. Now put $x=1$; thus *when* n *is even*

$$n=2^{\frac{n}{2}}\left(1-\cos\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\ldots\left(1-\cos\frac{n-4}{n}\pi\right)\left(1-\cos\frac{n-2}{n}\pi\right);$$

and by extracting the square root

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi\ldots\ldots(1).$$

The positive sign of the radical must be taken on the left-hand side, because the right-hand side is obviously positive.

Again, in (2) of Art. 314, divide both sides by $x-1$, and afterwards put $x=1$; thus *when* n *is odd*

$$n=2^{\frac{n-1}{2}}\left(1-\cos\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\ldots\left(1-\cos\frac{n-3}{n}\pi\right)\left(1-\cos\frac{n-1}{n}\pi\right);$$

and by extracting the square root,

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi\ldots\ldots(2).$$

Again, in (1) of Art. 315, put $x=1$; thus *when* n *is even*

$$2=2^{\frac{n}{2}}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{3\pi}{n}\right)\ldots\left(1-\cos\frac{n-3}{n}\pi\right)\left(1-\cos\frac{n-1}{n}\pi\right);$$

and by extracting the square root,

$$1=2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi\ldots\ldots(3).$$

Again, in (2) of Art. 315, put $x=1$; thus *when* n *is odd*

$$2=2^{\frac{n}{2}}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{3\pi}{n}\right)\ldots\left(1-\cos\frac{n-4}{n}\pi\right)\left(1-\cos\frac{n-2}{n}\pi\right);$$

and by extracting the square root,

$$1 = 2^{\frac{n-1}{2}} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \dots \sin \frac{n-4}{2n} \pi \sin \frac{n-2}{2n} \pi \dots\dots (4).$$

Four other results may apparently be deduced from the four formulæ of the two preceding articles by putting $x = -1$; but it will be found on trial that these results do not differ really from those already deduced. Thus, for example, in (1) of Art. 314, divide both sides by $x+1$, afterwards put $x=-1$, and extract the square root; thus *when* n *is even*

$$\sqrt{n} = 2^{\frac{n-1}{2}} \cos \frac{\pi}{n} \cos \frac{2\pi}{n} \dots \cos \frac{n-4}{2n} \pi \cos \frac{n-2}{2n} \pi;$$

this however is the same result as that in (1) of the present article, the factors on the right-hand side being merely differently arranged; for

$$\cos \frac{\pi}{n} = \sin \frac{n-2}{2n} \pi, \quad \cos \frac{2\pi}{n} = \sin \frac{n-4}{2n} \pi, \dots\dots$$

317. *To resolve* $x^{2n} - 2x^n \cos\theta + 1$ *into factors.*

If $\cos\theta = 1$ the expression becomes $(x^n-1)^2$, and if $\cos\theta = -1$ it becomes $(x^n+1)^2$; in these cases the resolution into factors is effected by what has already been given in Arts. 314 and 315, and we will therefore suppose these cases excluded from what follows. If we put

$$x^{2n} - 2x^n \cos\theta + 1 = 0,$$

we obtain $x^n = \cos\theta \pm \sqrt{(-1)} \sin\theta$; hence x is an n^{th} root of $\cos\theta + \sqrt{(-1)}\sin\theta$; the n^{th} roots are found from the expression $\cos\frac{2r\pi+\theta}{n} \pm \sqrt{(-1)} \sin \frac{2r\pi+\theta}{n}$ by ascribing integral values to r, for it is obvious from De Moivre's Theorem that the n^{th} power of the last expression is $\cos(2r\pi+\theta) \pm \sqrt{(-1)}\sin(2r\pi+\theta)$, and if r be an integer this reduces to $\cos\theta \pm \sqrt{(-1)}\sin\theta$. If we ascribe to r in succession the values, 0, 1, 2, ... $n-1$ in the expression $\cos\frac{2r\pi+\theta}{n} \pm \sqrt{(-1)} \sin\frac{2r\pi+\theta}{n}$ we obtain $2n$ *different* values for the expression. For if $r=p$ and $r=q$ could give the same value to the expression we should have

$$\cos\frac{2p\pi+\theta}{n}\pm\sqrt{(-1)}\sin\frac{2p\pi+\theta}{n}=\cos\frac{2q\pi+\theta}{n}\pm\sqrt{(-1)}\sin\frac{2q\pi+\theta}{n};$$

now by Art. 93 we cannot have $\cos\frac{2p\pi+\theta}{n}=\cos\frac{2q\pi+\theta}{n}$ and $\sin\frac{2p\pi+\theta}{n}=\sin\frac{2q\pi+\theta}{n}$; it is also impossible that

$$\cos\frac{2p\pi+\theta}{n}=\cos\frac{2q\pi+\theta}{n} \text{ and } \sin\frac{2p\pi+\theta}{n}=-\sin\frac{2q\pi+\theta}{n},$$

for that, by Art. 94, would require $\frac{2p\pi+\theta}{n}+\frac{2q\pi+\theta}{n}$ to be a multiple of 2π, so that θ would be a multiple of π, and this value of θ has been expressly excluded above. Thus we obtain $2n$ *different* values of x. Also the two factors

$$x-\cos\frac{2r\pi+\theta}{n}-\sqrt{(-1)}\sin\frac{2r\pi+\theta}{n},\quad x-\cos\frac{2r\pi+\theta}{n}+\sqrt{(-1)}\sin\frac{2r\pi+\theta}{n}$$

give by their product the real quadratic factor

$$\left(x-\cos\frac{2r\pi+\theta}{n}\right)^2+\sin^2\frac{2r\pi+\theta}{n}, \text{ that is, } x^2-2x\cos\frac{2r\pi+\theta}{n}+1.$$

Thus $x^{2n}-2x^n\cos\theta+1$

$$=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)\left(x^2-2x\cos\frac{4\pi+\theta}{n}+1\right)$$
$$\dots\left\{x^2-2x\cos\frac{(2n-4)\pi+\theta}{n}+1\right\}\left\{x^2-2x\cos\frac{(2n-2)\pi+\theta}{n}+1\right\}.$$

318. We shall now deduce some important results from the preceding general theorem. Suppose $x=1$; then

$$2(1-\cos\theta)=2^n\left(1-\cos\frac{\theta}{n}\right)\left(1-\cos\frac{2\pi+\theta}{n}\right)\left(1-\cos\frac{4\pi+\theta}{n}\right)\dots$$
$$\dots\left(1-\cos\frac{2n\pi-2\pi+\theta}{n}\right).$$

Let $\theta = 2n\phi$ and $\frac{\pi}{2n} = \alpha$; extract the square root; thus

$$\pm\sin n\phi = 2^{n-1}\sin\phi\sin(2\alpha+\phi)\sin(4\alpha+\phi)\ldots\ldots\sin(2n\alpha-2\alpha+\phi).$$

We shall now prove that the *upper* sign must always be taken on the left-hand side. First, suppose ϕ to lie between 0 and 2α; then every factor on the right-hand side is positive, and so is $\sin n\phi$. Next suppose ϕ to lie between 2α and 4α; then every factor on the right-hand side is positive *except the last*, and $\sin n\phi$ is negative. Next suppose ϕ to lie between 4α and 6α, then every factor on the right-hand side is positive *except the last two*, and $\sin n\phi$ is positive. By proceeding in this way we see that for every value of ϕ between 0 and $2n\alpha$, the upper sign must be taken, so that we have *for all values of* ϕ *between* 0 *and* π

$$\sin n\phi = 2^{n-1}\sin\phi\sin(2\alpha+\phi)\sin(4\alpha+\phi)\ldots\ldots\sin(2n\alpha-2\alpha+\phi).$$

We shall next shew that this formula is true for *all values* of ϕ; for suppose $\phi = m\pi+\psi$ where m is any integer, positive or negative, and ψ is between 0 and π; then we know that

$$\sin n\psi = 2^{n-1}\sin\psi\sin(2\alpha+\psi)\sin(4\alpha+\psi)\ldots\ldots\sin(2n\alpha-2\alpha+\psi);$$

but $\sin n\psi = \sin(n\phi - nm\pi) = \sin n\phi\cos nm\pi = (-1)^{nm}\sin n\phi$,

$$\sin\psi = \sin(\phi - m\pi) = \sin\phi\cos m\pi = (-1)^m\sin\phi,$$

$$\sin(2\alpha+\psi) = \sin(2\alpha+\phi-m\pi) = \sin(2\alpha+\phi)\cos m\pi = (-1)^m\sin(2\alpha+\phi),$$

and so on.

Substitute these values of $\sin n\psi$, $\sin\psi$, $\sin(2\alpha+\psi)$, ... in the formula which expresses $\sin n\psi$ in factors; then divide both sides by $(-1)^{nm}$ and we obtain the required formula for $\sin n\phi$, whatever may be the value of ϕ.

In the expression for $\sin n\phi$ change ϕ into $\phi+\alpha$; then $n\phi$ is changed into $n\phi + \frac{\pi}{2}$; hence

$$\cos n\phi = 2^{n-1}\sin(\phi+\alpha)\sin(\phi+3\alpha)\sin(\phi+5\alpha)\ldots\sin(2n\alpha-\alpha+\phi).$$

In the last result put $\phi = 0$; thus

$$1 = 2^{n-1} \sin \alpha \sin 3\alpha \sin 5\alpha \ldots\ldots \sin (2n\alpha - \alpha),$$

where $$\alpha = \frac{\pi}{2n}.$$

Again we have

$$\frac{\sin n\phi}{\sin \phi} = 2^{n-1} \sin (2\alpha + \phi) \sin (4\alpha + \phi) \ldots\ldots \sin (2n\alpha - 2\alpha + \phi);$$

now let ϕ diminish without limit; then since the limit of $\dfrac{\sin n\phi}{\sin \phi}$ is n we obtain

$$n = 2^{n-1} \sin 2\alpha \sin 4\alpha \sin 6\alpha \ldots\ldots \sin (2n\alpha - 2\alpha).$$

These two formulæ are sometimes useful.

319. The expression for $\sin n\phi$ in Art. 318 may be put into a different form; for

$$\sin (2n\alpha - 2\alpha + \phi) = \sin (\pi - 2\alpha + \phi) = \sin (2\alpha - \phi),$$
$$\sin (2n\alpha - 4\alpha + \phi) = \sin (\pi - 4\alpha + \phi) = \sin (4\alpha - \phi),$$

and so on.

Then by multiplying together the second factor and the last, the third and the last but one, and so on, we have

$$\sin n\phi = 2^{n-1} \sin \phi (\sin^2 2\alpha - \sin^2 \phi)(\sin^2 4\alpha - \sin^2 \phi) \ldots$$

It will be necessary to examine separately the cases when n is even and when n is odd.

First suppose n even; then the factor $\sin (n\alpha + \phi)$, that is, $\cos \phi$, will occur without any factor to multiply it; hence if n be even, we have

$$\sin n\phi = 2^{n-1} \sin \phi \cos \phi (\sin^2 2\alpha - \sin^2 \phi) (\sin^2 4\alpha - \sin^2 \phi) \ldots$$
$$\ldots \{\sin^2 (n-4) \alpha - \sin^2 \phi\} \{\sin^2 (n-2) \alpha - \sin^2 \phi\}.$$

Next suppose n odd; then we have

$$\sin n\phi = 2^{n-1} \sin \phi (\sin^2 2\alpha - \sin^2 \phi) (\sin^2 4\alpha - \sin^2 \phi) \ldots$$
$$\ldots \{\sin^2 (n-3) \alpha - \sin^2 \phi\} \{\sin^2 (n-1) \alpha - \sin^2 \phi\}.$$

Similarly from the formula

$$\cos n\phi = 2^{n-1} \sin (\phi + \alpha) \sin (\phi + 3\alpha) \sin (\phi + 5\alpha) \ldots \sin (2n\alpha - \alpha + \phi)$$

we obtain if n be even

$$\cos n\phi = 2^{n-1}(\sin^2 \alpha - \sin^2 \phi)(\sin^2 3\alpha - \sin^2 \phi) \dots$$
$$\dots \{\sin^2 (n-3)\alpha - \sin^2 \phi\}\{\sin^2 (n-1)\alpha - \sin^2 \phi\};$$

and if n be odd,

$$\cos n\phi = 2^{n-1} \cos \phi (\sin^2 \alpha - \sin^2 \phi)(\sin^2 3\alpha - \sin^2 \phi) \dots$$
$$\dots \{\sin^2 (n-4)\alpha - \sin^2 \phi\}\{\sin^2 (n-2)\alpha - \sin^2 \phi\}.$$

320. We can now resolve $\sin\theta$ and $\cos\theta$ into their factors. Suppose $n\phi = \theta$ and that n is odd; then by the preceding article

$$\sin\theta = 2^{n-1} \sin\frac{\theta}{n}\left(\sin^2 2\alpha - \sin^2\frac{\theta}{n}\right)\left(\sin^2 4\alpha - \sin^2\frac{\theta}{n}\right)\dots$$

divide both sides by $\sin\frac{\theta}{n}$, and then diminish θ indefinitely; since the limit of $\sin\theta \div \sin\frac{\theta}{n}$ is n we obtain

$$n = 2^{n-1} \sin^2 2\alpha \sin^2 4\alpha \dots;$$

therefore by division,

$$\sin\theta = n \sin\frac{\theta}{n}\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2 2\alpha}\right)\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2 4\alpha}\right)\dots.$$

Now suppose n to increase without limit; then since $\alpha = \frac{\pi}{2n}$ the limit of $\frac{\sin\frac{\theta}{n}}{\sin 2\alpha}$ is $\frac{\theta}{\pi}$, the limit of $\frac{\sin\frac{\theta}{n}}{\sin 4\alpha}$ is $\frac{\theta}{2\pi}$, and so on; thus finally,

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\dots$$

We shall obtain the same result if we begin by supposing n even.

Similarly we may shew that

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\dots.$$

321. In the same way as $x^n - 2x^n \cos\theta + 1$ was decomposed in Art. 317 we may decompose $x^{2n} - 2x^n a^n \cos\theta + a^{2n}$, and each quadratic factor of the last expression will be of the form $x^2 - 2xa\cos\dfrac{2r\pi + \theta}{n} + a^2$, where r is an integer; and all the factors are found by giving to r in succession the values 0, 1, 2, ... $n-1$. And $\cos\dfrac{2(n-1)\pi+\theta}{n} = \cos\dfrac{2\pi-\theta}{n}$, $\cos\dfrac{2(n-2)\pi+\theta}{n} = \cos\dfrac{4\pi-\theta}{n}$, and so on; thus all the factors will be found if we take $x^2 - 2xa\cos\dfrac{2r\pi\pm\theta}{n} + a^2$, and use both signs and give to r in succession the values 0, 1, 2, ... up to $\dfrac{n-1}{2}$ if n be odd, and up to $\dfrac{n}{2}$ if n be even; in the latter case when $r = \dfrac{n}{2}$ we must take only *one* factor $x^2 - 2xa\cos\dfrac{n\pi+\theta}{n} + a^2$.

Now suppose $x = 1 + \dfrac{z}{2n}$, and $a = 1 - \dfrac{z}{2n}$; thus

$$\left(1+\frac{z}{2n}\right)^{2n} - 2\left(1-\frac{z^2}{4n^2}\right)^n \cos\theta + \left(1-\frac{z}{2n}\right)^{2n}$$

is the expression to be decomposed into factors; and the general form of the factors is

$$\left(1+\frac{z}{2n}\right)^2 - 2\left(1-\frac{z^2}{4n^2}\right)\cos\frac{2r\pi\pm\theta}{n} + \left(1-\frac{z}{2n}\right)^2,$$

that is, $$2\left(1+\frac{z^2}{4n^2}\right) - 2\left(1-\frac{z^2}{4n^2}\right)\cos\frac{2r\pi\pm\theta}{n},$$

that is, $$4\sin^2\frac{2r\pi\pm\theta}{2n}\left(1+\frac{z^2}{4n^2}\cot^2\frac{2r\pi\pm\theta}{2n}\right).$$

Suppose n to increase indefinitely; then

$$\left(1+\frac{z}{2n}\right)^{2n} = e^z, \quad \left(1-\frac{z}{2n}\right)^{2n} = e^{-z}, \text{ (\textit{Algebra}, Art. 552)},$$

also $$\frac{z^2}{4n^2}\cot^2\frac{2r\pi\pm\theta}{2n}=\frac{z^2}{(2r\pi\pm\theta)^2};$$

and by putting $z=0$ we obtain

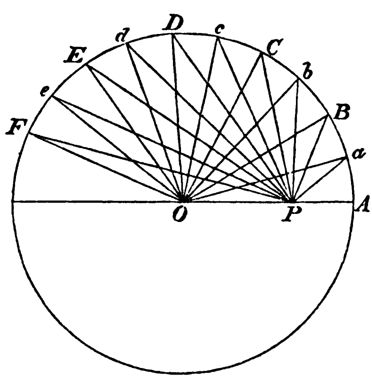
$$4\sin^2\frac{\theta}{2}=4\sin^2\frac{\theta}{2n}\,4\sin^2\frac{2\pi\pm\theta}{2n}\,4\sin^2\frac{4\pi\pm\theta}{2n}\dots;$$

thus finally

$$e^z-2\cos\theta+e^{-z}=4\sin^2\frac{\theta}{2}\left\{1+\frac{z^2}{\theta^2}\right\}\left\{1+\frac{z^2}{(2\pi\pm\theta)^2}\right\}\left\{1+\frac{z^2}{(4\pi\pm\theta)^2}\right\}\dots$$

Other examples of a similar kind may be seen in the sixth chapter of the third volume of the treatise on the *Differential and Integral Calculus* by Lacroix.

322. *De Moivre's property of the Circle.* Let O be the centre of a circle, P any point within it or without it; divide the whole



circumference into n equal arcs BC, CD, DE, ..., beginning at any point B, and join O and P with the points of division B, C, D, ... Let $POB=\theta$; then will

$$OP^{2n}-2OP^n\,.\,OB^n\cos n\theta+OB^{2n}=PB^2\,.\,PC^2\,.\,PD^2\dots \text{ to } n \text{ factors.}$$

For $$PB^2 = OP^2 - 2OP\,.\,OB\cos\theta + OB^2,$$

$$PC^2 = OP^2 - 2OP\,.\,OC\cos\left(\theta + \frac{2\pi}{n}\right) + OC^2,$$

$$PD^2 = OP^2 - 2OP\,.\,OD\cos\left(\theta + \frac{4\pi}{n}\right) + OD^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and the radii OB, OC, OD are all equal.

Thus, by Arts. 317 and 321, the product of all the terms on the right-hand side of these equations is

$$OP^{2n} - 2OP^n\,.\,OB^n\cos n\theta + OB^{2n};$$

this proves the proposition.

The particular case when P is on the circumference may be noticed; then

$$2OB^n\sin\frac{n\theta}{2} = PB\,.\,PC\,.\,PD\ldots \text{ to } n \text{ factors.}$$

Cotes's properties of the Circle. These are particular cases of De Moivre's property of the circle.

Let OP produced if necessary meet the circle in A, and suppose $AB = BC = \frac{2\pi}{n}$; then $n\theta = 2\pi$. Thus we obtain

$$(OP^n - OB^n)^2 = PB^2\,.\,PC^2\,.\,PD^2\ldots \text{ to } n \text{ factors;}$$

therefore $$OP^n \sim OB^n = PB\,.\,PC\,.\,PD\ldots \text{ to } n \text{ factors.}$$

Again, let the arcs AB, BC, ... be bisected in a, b, ... ; then by the theorem just proved,

$$OP^{2n} \sim OB^{2n} = Pa\,.\,PB\,.\,Pb\,.\,PC\ldots \text{ to } 2n \text{ factors;}$$

therefore by division,

$$OP^n + OB^n = Pa\,.\,Pb\,.\,Pc\ldots \text{ to } n \text{ factors.}$$

323. It is usual in works on Trigonometry to give a brief though unsatisfactory demonstration of the results of Article 320 in the following manner.

Since $\sin\theta$ vanishes when $\theta = 0$, or $\pm\pi$, or $\pm 2\pi$, ... *it follows that* $\sin\theta$ *must be divisible by* θ, $\theta+\pi$, $\theta-\pi$, $\theta+2\pi$, $\theta-2\pi$,... ; *therefore we may assume that*

$$\sin\theta = A\theta(\theta-\pi)(\theta+\pi)(\theta-2\pi)(\theta+2\pi)(\theta-3\pi)(\theta+3\pi)\ldots$$

where A *is some quantity independent of* θ; thus we may suppose

$$\sin\theta = a\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

where a is also some quantity independent of θ. Divide both sides by θ and then suppose $\theta = 0$; thus $a = 1$, and consequently

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Again, since $\cos\theta$ vanishes when $\theta = \pm\frac{\pi}{2}$, or $\pm\frac{3\pi}{2}$, ... *it follows that* $\cos\theta$ *must be divisible by* $\theta-\frac{\pi}{2}$, $\theta+\frac{\pi}{2}$, $\theta-\frac{3\pi}{2}$, $\theta+\frac{3\pi}{2}$, ... *therefore we may assume that*

$$\cos\theta = A\left(\theta-\frac{\pi}{2}\right)\left(\theta+\frac{\pi}{2}\right)\left(\theta-\frac{3\pi}{2}\right)\left(\theta+\frac{3\pi}{2}\right)\left(\theta-\frac{5\pi}{2}\right)\left(\theta+\frac{5\pi}{2}\right)\ldots$$

where A *is some quantity independent of* θ; thus we may suppose

$$\cos\theta = a\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

where a is also some quantity independent of θ; and by putting $\theta = 0$ we find $a = 1$; thus

$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

The portions of the preceding investigations which are printed in italics involve assumptions which cannot be considered legitimate.

324. It has been stated in Art. 169, that the tables of the logarithms of Trigonometrical functions can be calculated without the use of the tables of the Natural functions; we will here briefly indicate how this may be effected. We have

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots;$$

put $\frac{m}{n}\frac{\pi}{2}$ for θ and take logarithms; thus

$$\log\sin\frac{m}{n}\frac{\pi}{2} = \log\frac{m}{n} + \log\frac{\pi}{2} + \log\left(1-\frac{m^2}{4n^2}\right)$$
$$+\log\left(1-\frac{m^2}{2^2 4n^2}\right) + \log\left(1-\frac{m^2}{3^2 4n^2}\right) + \dots$$

The terms in the last line may be expanded by Art. 145 in series which will converge with sufficient rapidity; thus we shall have if μ denote the modulus

$$\log\sin\frac{m}{n}\frac{\pi}{2} = \log\pi + \log m + \log(2n+m) + \log(2n-m) - 3(\log 2 + \log n)$$
$$-\mu\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\dots\right)\frac{m^2}{n^2}$$
$$-\frac{\mu}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\dots\right)\frac{m^4}{n^4}$$
$$-\frac{\mu}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\dots\right)\frac{m^6}{n^6}$$
$$-\dots\dots\dots\dots$$

Similarly we may find $\log\cos\frac{m}{n}\frac{\pi}{2}$. (Airy's *Trigonometry.*)

325. We will now make a few remarks on the symbol $\sqrt{(-1)}$, which has been used very often throughout the latter portion of this book. We may consider that the symbol has been used in an *experimental* manner, and many results have been obtained by means of it; the point now to be considered is how far these results can be received as true.

In the first place, some of the results obtained by using the symbol $\sqrt{(-1)}$ may be shewn to be true by other methods; thus, for example, the values obtained for $\sin n\theta$ and $\cos n\theta$ in Art. 269 may be verified by *induction.*

Again, the following example will shew how in some cases a strict demonstration may be obtained even with the use of the symbol $\sqrt{(-1)}$. Let n be a positive integer, and suppose it required to expand $\cos^n \theta$ in terms of cosines of multiples of θ; we may proceed as we did in Art. 280, supposing x to stand for $e^{\theta\sqrt{(-1)}}$. Now we know that

$$(e^y + e^{-y})^n = e^{ny} + e^{-ny} + n\{e^{(n-2)y} + e^{-(n-2)y}\} + \frac{n(n-1)}{1.2}\{e^{(n-4)y} + e^{-(n-4)y}\} + \ldots$$

thus

$$2^{n-1}\left\{1 + \frac{y^2}{1.2} + \frac{y^4}{\lfloor 4} + \frac{y^6}{\lfloor 6} + \ldots\right\}^n$$

$$= 1 + \frac{n^2y^2}{1.2} + \frac{n^4y^4}{\lfloor 4} + \frac{n^6y^6}{\lfloor 6} + \ldots$$

$$+ n\left\{1 + \frac{(n-2)^2y^2}{1.2} + \frac{(n-2)^4y^4}{\lfloor 4} + \frac{(n-2)^6y^6}{\lfloor 6} + \ldots\right\}$$

$$+ \ldots\ldots$$

Now this is true for all values of y, that is, if all the operations indicated be performed, the two members of the equation are *identically* equal. We may therefore put $-\theta^2$ instead of y^2, and the result will still be true. Thus

$$2^{n-1}\left\{1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{\lfloor 4} - \ldots\right\}^n = 1 - \frac{n^2\theta^2}{1.2} + \frac{n^4\theta^4}{\lfloor 4} - \ldots$$

$$+ n\left\{1 - \frac{(n-2)^2\theta^2}{1.2} + \frac{(n-2)^4\theta^4}{\lfloor 4} - \ldots\right\}$$

$$+ \ldots\ldots$$

Thus $2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \ldots\ldots$

(Airy's *Trigonometry*.)

Finally, the student may be informed that a theory has been constructed which offers a complete explanation of the symbol $\sqrt{(-1)}$, and thus enables us to obtain rigid demonstrations by the use of this symbol. It is not consistent with the plan of the present work to give any account of this theory; the student, however, is recommended hereafter to read the *Trigonometry and Double Algebra* of Professor De Morgan.

EXAMPLES.

1. Find the sums of the following infinite series:

(1) $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots$

(2) $\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$

when $n=2$ and when $n=4$.

2. If $a=\frac{\pi}{4n}$, shew that

$$\sin a \sin 5a \sin 9a \dots\dots \sin (4n-3)\, a = 2^{-n+\frac{1}{2}}.$$

3. A polygon of n sides inscribed in a circle is such that its sides subtend angles a, $2a$, $3a$, ... na at the centre; shew that the ratio of the area of this polygon to the area of the regular inscribed polygon of n sides is equal to that of $\sin\frac{na}{2}$ to $n\sin\frac{a}{2}$.

4. The product of all the lines that can be drawn from one of the angles of a regular polygon of n sides inscribed in a circle whose radius is a to all the other angular points is na^{n-1}.

5. If p_1, p_2,...p_{2n-1}, p_{2n} be the perpendiculars drawn from any point in the circumference of a circle of radius a on the sides of a regular circumscribing polygon of $2n$ sides, shew that

$$p_1 p_3 p_5 \cdots p_{2n-1} + p_2 p_4 \cdots p_{2n} = \frac{a^n}{2^{n-2}}.$$

6. A polygon is described about a circle touching it at the angular points of an inscribed polygon; the product of the perpendiculars drawn to the several sides of the inscribed polygon from

any point in the circumference of the circle is equal to the product of the perpendiculars drawn from the same point to the several sides of the circumscribed polygon.

7. Prove that $\sin\theta\cos\frac{\theta}{2} = 8\sin\frac{\theta}{2}\sin^2\frac{\pi-\theta}{4}\sin^2\frac{\pi+\theta}{4}$.

8. Prove that

$$\left(\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{3} = \left(\tan^2\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\cot\frac{2\theta}{3}.$$

9. Prove that

$$\tan 3\theta - \tan 2\theta - \tan\theta = \tan 3\theta\tan 2\theta\tan\theta.$$

10. Find x from the equation

$$\tan^3 x + \cot^3 x = m^3 - 3m.$$

11. The circumference of a circle is divided into $2n$ equal parts in the points A, P, Q,.... Tangents are drawn at the points A, P, Q, ... and perpendiculars OA, OB, OC, ... are let fall upon them from O the extremity of the diameter OA. Shew that

$$OA^2 + OB^2 + OC^2 + \ldots\ldots = 3n\,(\text{radius})^2.$$

12. ACB is a quadrant; AP, AQ, AR are three arcs in ascending order of magnitude, each being less than AB, and their sum equal to twice AB; radii CP, CQ, CR are produced to meet the tangent at A in p, q, r, and a triangle is formed with Ap, Aq, Ar. Find the condition that this may be possible, and the inferior limit of Aq and the superior limit of Ap. Prove also that in all such triangles the radii of the inscribed and circumscribed circles are inversely proportional.

13. ABC is a right-angled triangle, C being the right angle, E is the point in which the inscribed circle touches BC, and F the point in which the circle drawn to touch AB and the sides CA, CB produced meets CA; shew that if EF be joined the triangle FEC is half the triangle ABC.

14. Through the angular points of a triangle lines are drawn bisecting the exterior angles. If S be the area of the original triangle and S' that of the new triangle, shew that

$$S' = \frac{1}{2} S \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}.$$

15. $ABCD$ is a horizontal straight line. From a point immediately above D the known distances AB and BC are observed to subtend the same angle α. If $AB = a$ and $BC = b$, shew that the height of the observer's position above D is

$$\frac{2ab(a+b)\tan\alpha}{(a-b)^2 + (a+b)^2 \tan^2\alpha}.$$

16. If in any arc not greater than a quadrant a point be taken, and from this point two lines be drawn, one to the extremity of the arc, the other perpendicular to its chord and terminated by it, prove that the sum of these two lines is less than the chord of the arc.

17. Suppose α the angle of elevation of a cloud, β the angle of depression of the image of the cloud seen by reflection from a lake, h the height of the observer's eye above the lake, then the height of the cloud is

$$\frac{h \sin(\beta + \alpha)}{\sin(\beta - \alpha)}.$$

18. At noon a person standing on a cliff h feet above the level of the sea, observes the altitude of a cloud in the plane of the meridian to be α and the angle of depression of its shadow on the surface of the water to be β; shew that, if γ be the sun's altitude at the time of observation, the height of the cloud above the surface of the water will be

$$\frac{h \sin\gamma \sin(\alpha + \beta)}{\sin\beta \sin(\gamma + \alpha)},$$

the sun being *behind* the observer when he is looking at the cloud.

ANSWERS.

I. II. III. IV. V.

I. page 6. 1. $18^\circ, 27^\circ$. 2. $15^\circ, 45^\circ$. 3. $30^\circ, 15^\circ$.

4. $\cdot 00945$. 5. $\frac{50n^\circ}{77}, \frac{27n^\circ}{77}$. 6. $2\frac{1}{2}$. 7. 3°.

8. One polygon has 8 sides, and the other 12 sides; so that an angle of the first is $\frac{3}{2}$ of a right angle, and an angle of the second $\frac{5}{3}$ of a right angle. 10. The ratio is that of 5 to 162.

II. pages 13, 14. 2. $\frac{3}{40} \times \frac{180}{\pi}$. 3. $\pi \times \cdot 00505$. 4. $27^\circ, 9^\circ, 18^\circ$.

5. $\frac{5\pi}{32}$, $28^\circ \cdot 125$, $31^g \cdot 25$. 6. $40^\circ, 60^\circ, 80^\circ$. 7. $30^\circ, 60^\circ, 90^\circ$.

III. pages 22, 23. 6. $\theta = \frac{\pi}{3}$. 7. $\theta = 0$ or $\frac{\pi}{2}$. 8. $\theta = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

9. $\theta = \frac{\pi}{3}$. 10. $\theta = \frac{\pi}{6}$ or $\frac{\pi}{4}$. 11. $A = 45^\circ$; $B = 15^\circ$.

IV. page 41. 1. The same as for an angle of 225°.

2. The same as for an angle of 330°. 3. The same as for an angle of 210°. 4. The same as for an angle of 300°.

5. $45^\circ, 225^\circ, 405^\circ, 585^\circ, 765^\circ$.

6. $45^\circ, 135^\circ, 225^\circ, 315^\circ, 405^\circ, 495^\circ, 585^\circ, 675^\circ, 765^\circ, 855^\circ$.

7. $0, 1 - \frac{1}{\sqrt{2}}, 1, 1 + \frac{1}{\sqrt{2}}, 2$. 8. $\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}$.

9. We have $\sin\theta = -\cos\theta$; therefore $\theta = 135^\circ$, &c.

10. $\cos\theta = -\frac{1}{2}$; therefore $\theta = 120^\circ$, &c. 14. No.

V. page 49. 1. $n\pi + \frac{\pi}{4}$. 2. $(2n + \frac{1}{2})\pi$. 3. $2n\pi$.

4. $2n\pi \pm \frac{2\pi}{3}$. 5. $n\pi \pm \alpha$. 6. $n\pi \pm \frac{\pi}{3}$. 7. $n\pi \pm \alpha$.

8. $n\pi \pm \frac{\pi}{4}$. 9. $n\pi \pm \alpha$. 10. $n\pi \pm \frac{\pi}{6}$. 12. $2n\pi + \frac{7\pi}{6}$.

VI. page 61. 31. $\theta = n\pi \pm \frac{\pi}{6}$. 32. $\frac{5\theta}{2} = n\pi$ or $\frac{3\theta}{2} = (n + \frac{1}{2})\pi$.

33. $3\theta = n\pi$ or $4\theta = 2n\pi \pm \frac{\pi}{3}$. 34. $\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}$.

35. $\theta = n\pi$ or $n\pi \pm \frac{\pi}{6}$. 36. $2\theta = (n + \frac{1}{2})\pi$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$.

37. $2\theta = n\pi$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$. 38. $2\theta = n\pi + (-1)^n \frac{\pi}{6}$.

39. $\theta = (n + \frac{1}{2})\pi$ or $4\theta = n\pi + (-1)^n \frac{\pi}{6}$. 40. $\theta + \frac{\pi}{4} = n\pi \pm \frac{\pi}{3}$.

VII. pages 68—70. 2. $2 \cos \frac{A}{2} = \sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)}$.

3. $2 \sin \frac{A}{2} = -\sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)}$.

4. $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$. 5. $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

6. $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$. 10. $\frac{1}{2}$. 11. $-\frac{\sqrt{3} - 1}{2\sqrt{2}}$.

12. $\sin A = \pm\frac{4}{5}$, $\cos A = \pm\frac{3}{5}$; or $\sin A = \pm\frac{3}{5}$, $\cos A = \mp\frac{4}{5}$.

13. $\sqrt{3} - 2$. 26. ± 1. 27. 4. 28. $-\frac{1}{2}$.

VIII. pages 77—79. Example 20 may be deduced from example 16 by changing A into $\frac{1}{2}(\pi - A)$ and making similar changes for B and C; example 21 may be deduced from example 17 in the same way.

38. $$\frac{\cos x - \cos \alpha}{\cos x - \cos \beta} = \frac{\sin^2 \alpha \cos \beta}{\sin^2 \beta \cos \alpha};$$

therefore $$\cos x = \frac{\sin^2 \beta \cos^2 \alpha - \sin^2 \alpha \cos^2 \beta}{\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta} = \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta};$$

then find $\frac{1 - \cos x}{1 + \cos x}$. 39. $\frac{\tan^2 \theta}{\tan^2 \theta'} = \frac{\tan^2 \alpha}{\tan^2 \alpha'}$; that is,

$$\frac{\cos \beta - \cos \alpha}{\cos \alpha} \div \frac{\cos \beta - \cos \alpha'}{\cos \alpha'} = \frac{\tan^2 \alpha}{\tan^2 \alpha'};$$ therefore

$$\frac{\cos\beta-\cos\alpha}{\cos\beta-\cos\alpha'}=\frac{\sin^2\alpha\cos\alpha'}{\sin^2\alpha'\cos\alpha};\ \cos\beta=\frac{\sin^2\alpha'\cos^2\alpha-\sin^2\alpha\cos^2\alpha'}{\sin^2\alpha'\cos\alpha-\sin^2\alpha\cos\alpha'}$$

$$=\frac{\cos^2\alpha-\cos^2\alpha'}{\cos\alpha-\cos\alpha'-\cos\alpha\cos^2\alpha'+\cos\alpha'\cos^2\alpha}=\frac{\cos\alpha+\cos\alpha'}{1+\cos\alpha\cos\alpha'};$$

then find $\dfrac{1-\cos\beta}{1+\cos\beta}$. 48. Put $\dfrac{2\tan\frac{1}{2}\phi}{1-\tan^2\frac{1}{2}\phi}$ for $\tan\phi$; then solve the quadratic; thus we shall find

$$\tan\tfrac{1}{2}\phi=-\frac{(\cos\theta+\sin\theta')\pm(1+\sin\theta'\cos\theta)}{\sin\theta\cos\theta'};$$

the lower sign gives the required result. The upper sign gives

$$-\cot\frac{\theta}{2}\cot\left(\frac{\pi}{4}-\frac{\theta'}{2}\right).$$

52. By Example 23, page 77, we get $\cos A\cos B\cos C=0$, so that one of the three angles is a right angle.

IX. pages 90, 91.

5. $\tan(\theta-\phi)=\dfrac{(n-1)\tan\phi}{1+n\tan^2\phi}=\dfrac{n-1}{\{\sqrt{(n\tan\phi)}-\sqrt{\cot\phi}\}^2+2\sqrt{n}}$; the greatest value of this is when the first term of the denominator vanishes.

6. $2\sin\theta\sin^2\dfrac{\theta+\phi}{2}$.

8. The height in yards $=1760\times\tan 1'=\dfrac{1760\pi}{180.60}$ nearly.

9. Let x be the distance, $\dfrac{3}{x}=\tan\dfrac{1^\circ}{4}$; thus $\dfrac{3}{x}=\dfrac{\pi}{180\times 4}$ nearly.

10. We get $\sin A=\pm\frac{1}{2}\sqrt{(3-n)}$. 12. 6. 16. 8.

17. $\theta-\dfrac{\pi}{4}=2n\pi$. 18. $\theta+\dfrac{\pi}{3}=2n\pi\pm\dfrac{3\pi}{4}$. 19. $\dfrac{\pi}{2}-2\theta=2n\pi\pm\theta$.

20. $\theta=n\pi+\dfrac{3\pi}{4}$ or $2\theta=n\pi+(-1)^n\dfrac{\pi}{3}$. 21. $\dfrac{3\theta}{2}=n\pi$ or $\dfrac{\pi}{2}-\dfrac{\theta}{2}=2n\pi\pm\dfrac{3\theta}{2}$. 22. $\theta=n\pi+\dfrac{3\pi}{4}$ or $\sin 2\theta=2(\sqrt{2}-1)$.

23. $\theta=(2n+1)\dfrac{\pi}{2}$ or $n\pi\pm\dfrac{\pi}{4}$. 24. $\theta=(2n+1)\dfrac{\pi}{4}$.

25. $\theta = n\pi \pm \frac{\pi}{10}$ or $n\pi \pm \frac{3\pi}{10}$. 26. $\frac{\theta}{2} = n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

27. $\theta = \frac{n\pi}{8}$. 28. $\theta = n\pi \pm \frac{\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$. 29. $\theta = n\pi \pm \frac{\pi}{4}$.

30. $\theta = n\pi$ or $n\pi + \frac{3\pi}{4}$. 31. $\sin\frac{5\theta}{2} = 0$, or $\cos\theta = 0$, or $\cos\frac{\theta}{2} = 0$.

32. $\cos\theta + \sin 3\theta = 0$, that is, $\cos\theta = \cos\left(3\theta + \frac{\pi}{2}\right)$.

33. $2\theta = n\pi \pm \frac{\pi}{12}$. 34. $\sin\theta = -1$, or $\sin\frac{\theta}{2} = 0$, or $\tan\frac{\theta}{2} = 2$.

35. $2\theta = (2n+1)\frac{\pi}{2}$, or $7\theta = n\pi + (-1)^n\frac{\pi}{6}$.

It should be remarked that answers may be given under apparently different forms; thus, for example, suppose we have to solve the equation $\sin 2\theta = \cos\theta$, or $2\sin\theta\cos\theta = \cos\theta$,

this gives $\theta = 2n\pi \pm \frac{\pi}{2}$ and $\theta = n\pi + (-1)^n\frac{\pi}{6}$;

but we may write the equation $\cos\left(\frac{\pi}{2} - 2\theta\right) = \cos\theta$;

therefore $\frac{\pi}{2} - 2\theta = 2n\pi \pm \theta$.

X. pages 100—104. 1. $\frac{21}{2}$. 2. $243\sqrt[3]{9} = (\sqrt{3})^{\frac{34}{3}}$. 3. $7; -4; -\frac{1}{2}$.

4. 1·06. 6. $3; -1$. 10. $\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \ldots = e^{-1}$.

12. $2x - a = 2n\pi \pm \frac{\pi}{3}$. 13. $x = a\cos(\alpha - \beta)$ or $-a\cos(\alpha + \beta)$.

14. $x = 2n\pi \pm \frac{\pi}{5}$ or $2n\pi \pm \frac{3\pi}{5}$. 15. $\cos(x+1)\alpha = \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$.

16. $x = \sec\left(\alpha - \frac{\beta}{2}\right)$ or $-2\cos\frac{\beta}{2}\sec\alpha$. 17. We can get $\sin 2^x\alpha = \sin 3\alpha$. 18. $\sin\frac{\alpha - \theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}$; this gives $\tan\frac{\theta}{2}$.

19. $\theta = m\pi + \frac{\pi}{2}$ or $(n-1)\theta = 2m\pi \pm \frac{\pi}{3}$.

20. $\cos\theta = 0$, or $\sin\frac{\theta}{2} = 0$, or $\cos\frac{5\theta}{2} = 0$. 22. $\frac{\pi}{16}$.

23. $n = 2$. 24. $\sin^3\frac{\theta - a}{2}\sin\frac{3\theta + a}{2} = 0$.

31. Write for x successively $\frac{\pi}{2} - x$ and $\frac{\pi}{2} + x$.

34. By Art. 114, $\tan^2 A + \tan^2 B + \tan^2 C = 1 + \frac{1}{2}(\tan A - \tan B)^2 + \frac{1}{2}(\tan B - \tan C)^2 + \frac{1}{2}(\tan C - \tan A)^2$.

36. $\cot B + \cot C - \operatorname{cosec} A$

$$= \frac{\sin(B+C)}{\sin B \sin C} - \frac{1}{\sin A} = \frac{\sin^2 A - \sin B \sin C}{\sin A \sin B \sin C}, \text{ \&c.}$$

37. If $A + B + C = 180°$, we have

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C \ldots\ldots\ldots\ldots (1).$$

Thus, if A, B, C are all *acute*, the sum of the squares of the cosines is *less* than unity. Hence if we require the sum of the squares of the cosines to be *equal* to unity, one or more of the acute angles must be diminished, so that their sum will then be less than 180°.

38. From the value of $\sin(A + B + C)$, given in Art. 113, it will follow that

$$\sin A + \sin B + \sin C - \sin(A + B + C)$$
$$= \sin A(1 - \cos B\cos C) + \sin B(1 - \cos A \cos C) + \sin C(1 - \cos A \cos B) + \sin A \sin B \sin C\,;$$

and every term of this expression is positive.

39. $e^{-\frac{a^2}{2}}$. 40. zero. 41. It depends on $(1 - \cos\theta)^2(1 + 2\cos\theta)$ being greater than zero.

XII. pages 141—145. 3. $\tan\theta = \frac{-\sin a \pm \sin\beta}{\cos a \cos\beta}$.

4. 1; $\frac{1}{2}$. 5. $\cot\frac{\theta}{2} - \cot\theta = \frac{1}{\sin\theta}$.

6. $\sqrt{\frac{c-1}{c+1}}$. 7. $a^2 = b^2$. 8. $a^2 + b^2 + \frac{c^2}{\sin^2\theta} = 1$.

11. $a^2+b^2-2c=2.$ 12. $x^2+y^2=a^2\left(1+\frac{y^2}{b^2}\right).$

13. $\cot c=\frac{1}{a}-\frac{1}{b}.$ 14. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ 15. $b^2=a^2-2ac\cos 2\phi+c^2.$

19. $(mn)^{\frac{2}{3}}\{m^{\frac{2}{3}}+n^{\frac{2}{3}}\}=1.$ 20. $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$

29. $\sin\theta\sin\phi=\sin\alpha\sin\beta,$

therefore $$4\sin^2\frac{\theta}{2}-4\sin^4\frac{\theta}{2}=\frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi},$$

therefore $$4\sin^4\frac{\theta}{2}-4\sin^2\frac{\theta}{2}+1=1-\frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi};$$

and $$\sin^2\phi=\frac{\cot^2\frac{\alpha}{2}}{\cot^2\frac{\alpha}{2}+\cos^2\beta};$$ therefore

$$2\sin^2\frac{\theta}{2}-1=\pm\sqrt{\left\{1-4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2}+\cos^2\beta\right)\sin^2\beta\right\}};$$

this reduces to $$2\sin^2\frac{\theta}{2}-1=\pm\left(1-2\sin^2\frac{\alpha}{2}\sin^2\beta\right).$$

30. n must lie between -2 and -1 or between 1 and 2.

31. By Art. 114 we may suppose $x=\tan A$, $y=\tan B$, $z=\tan C$, where $A+B+C=180°$. Therefore $2A+2B+2C=360°$; and $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$. This gives the required result.

32. $v\sin c=\sin z=-\sin x\cos y-\cos x\sin y$

$=-v\sin a\cos y-v\sin b\cos x$, or $\sin a\cos y=-\sin c-\sin b\cos x$;

and $\sin a\sin y=\sin b\sin x$; square and add, thus

$\sin^2 a=\sin^2 b+\sin^2 c+2\sin b\sin c\cos x$; therefore

$$\cos x=\frac{\sin^2 a-\sin^2 b-\sin^2 c}{2\sin b\sin c}.$$

Similarly $\cos y$ and $\cos z$ may be found.

33. $e^{-\frac{\alpha^2}{\beta^2}}.$

37. We have universally

$$\sin^2(A+B) = \sin^2 A + \sin^2 B + 2\sin A \sin B \cos(A+B) \ldots\ldots (1);$$

also in the present case $\sin^2 A + \sin^2 B = \cos^2 C \ldots\ldots (2)$.

If $A+B$ is greater than $90°$, then *a fortiori* $A+B+C$ is so also.

If $A+B$ is less than $90°$, then $\sin^2(A+B)$ is greater than $\sin^2 A + \sin^2 B$ by (1), that is, greater than $\cos^2 C$ by (2);

therefore $A+B$ is greater than $90° - C$.

XIII. pages 152—155.

5. Let $\frac{\pi}{14} = \alpha$ so that the angles of the triangle are 2α, 4α and 8α.

Then the ratio of the greatest side to the perimeter

$$= \frac{\sin 8\alpha}{\sin 2\alpha + \sin 4\alpha + \sin 8\alpha} = \frac{\sin 8\alpha}{\sin 2\alpha + \sin 4\alpha + \sin 6\alpha}$$

$$= \frac{2\sin 4\alpha \cos 4\alpha}{2\sin 3\alpha \cos \alpha + 2\sin 3\alpha \cos 3\alpha} = \frac{\sin 4\alpha}{\cos\alpha + \cos 3\alpha}$$

$$= \frac{2\sin 2\alpha \cos 2\alpha}{2\cos 2\alpha \cos\alpha} = 2\sin\alpha.$$

8 $\frac{\sin 2\theta + \sin 4\theta}{\sin 3\theta} = \frac{a+c}{b}$, therefore $2\cos\theta = \frac{a+c}{b}$.

21. $\sin\theta + \sin\phi = 2\sin(\theta+\phi)$; therefore $\cos\frac{\theta-\phi}{2} = 2\cos\frac{\theta+\phi}{2}$;

therefore $\cos\frac{\theta}{2}\cos\frac{\phi}{2} = 3\sin\frac{\theta}{2}\sin\frac{\phi}{2}$;

therefore $\left(1-\sin^2\frac{\theta}{2}\right)\left(1-\sin^2\frac{\phi}{2}\right) = 9\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}$;

therefore $8\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2} = 1 - \sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}$;

therefore $16\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2} = 2 - 2\sin^2\frac{\theta}{2} - 2\sin^2\frac{\phi}{2} = \cos\theta + \cos\phi$.

Or thus, $\cos\theta = \frac{a^2+b^2-c^2}{2ab}$ and $b = \frac{a+c}{2}$;

therefore $\cos\theta = \frac{a-c}{a} + \frac{b}{2a} = \frac{5a-3c}{4a}$;

similarly $\cos\phi = \frac{5c-3a}{4c}$.

37. This will follow from examples 20 and 21 of Chapter VIII.

40. We have to shew that $(b+c-a)(c+a-b)(a+b-c)$ is less than abc except when $a=b=c$. By squaring, this amounts to shewing that $\{a^2-(c-b)^2\}\{b^2-(a-c)^2\}\{c^2-(a-b)^2\}$ is less than $a^2b^2c^2$; and each factor on the left-hand side is less than the corresponding factor on the right side except when $a=b=c$.

XIV. pages 166—169. 1. $A = 30°$ or $150°$. 2. $30°, 90°$.

3. $45°, 60°, 75°$. 4. The triangle is impossible.

5. $B = 90°$, $C = 72°$, $c = 4\sqrt{(5+2\sqrt{5})}$. 6. $B = 45°$ or $135°$.

7. From Art. 235 we have $c + c' = 2b\cos A$ and $cc' = b^2 - a^2$.

8. $b^2 \sin A \cos A$. 11. No; the triangle is right angled.

12. We get $\sin\theta = \frac{2\sqrt{(ab)}}{c}\sin\frac{1}{2}C$;

also $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$.

13. $c^2 = a^2 + b^2 - 2ab\cos C = (a-b)^2 + 4ab\sin^2\frac{1}{2}C$; &c.

14. 9·6733937. 15. 132° 34′ 32″. 16. 55° 46′ 16″.

17. 78° 27′ 47″. 18. 119° 26′ 51″; 5° 33′ 9″.

19. 69° 10′ 10″; 46° 37′ 50″. 20. 116° 33′ 54″; 26° 33′ 54″.

21. 82° 10′ 50″; 50° 24′ 10″. 22. 124° 48′ 59″; 33° 11′ 1″.

24. 48° 11′ 23″; 58° 24′ 43″; 73° 23′ 54″.

25. $\cos A = \frac{3}{6953}$, therefore $\sin^2\frac{A}{2} = \frac{3475}{6953}$; $B = 1' 29''$.

26. 70° 53′ 36″; 49° 6′ 24″. 27. 38° 12′ 47″; 21° 47′ 13″.

28. 26° 33′ 54″. 29. 69° 49′ 35″; 50° 10′ 25″. 30. 30° or 15°.

XV. pages 175—181. In order to solve some of these examples the student must be acquainted with the Mariner's Compass. In the Mariner's Compass the circumference of a circle is divided into thirty-two equal parts, so that each part subtends at the centre of the circle an angle of $\frac{360}{32}$ degrees, that is, an angle of $11\frac{1}{4}°$. The following names are assigned to the points of division of the circumference, North, North by East, North North East, North East by North, North East, North East by East, East North East, East by North, East, East by South, East South East, South East by East, South East, South East by South, South South East, South by East, South, South by West, South South West, South West by South, South West, South West by West, West South West, West by South, West, West by North, West North West, North West by West, North West, North West by North, North North West, North by West.

1. $880(3+\sqrt{3})$. 2. $\frac{100}{\sqrt{3}}$ yards. 8. The distance of the eye from the foot of the tower $= b\left(\frac{a+b-2h}{a-b}\right)^{\frac{1}{2}}$. 10. Let x denote the required height; then eliminate θ between

$$x = b\tan\theta, \quad a + x = b\tan(\theta+\gamma).$$

11. $10\sqrt{(115)}$ feet; neglecting the height of the observer's eye from the ground. 12. $40\sqrt{3}$ feet.

13. Height $40\sqrt{6}$ feet; distance $40\{\sqrt{(14)}+\sqrt{2}\}$ feet.

18. $8+4\sqrt{2}$ miles per hour.

22. Let h' be the height of the higher hill, h of the lower; then

$$h = \frac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)}, \text{ and } \frac{h'}{h} = \frac{h'\cot\alpha' - c}{h\cot\beta + 1}.$$

23. $180\sqrt{3}$ feet. 25. $B = 60°$ or $120°$; approximate error $6''$.

26. $\frac{2c\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha+\beta)}$.

29. $\sqrt{(a^2+2ab\cos\alpha+b^2)}$. 30. Suppose both lines OC and $O'C$ to fall within the angle ACB. Let $AC = a$, $ACO = \phi$; then from the triangles ACO and BCO we get

$$OC = \frac{a \sin(\phi + \alpha)}{\sin \alpha} = \frac{a \cos(\phi - \beta)}{\sin \beta}.$$

Hence $\tan \phi$ is known and then $\sin \phi$ and $\cos \phi$. Thus we shall get $OC^2 = \dfrac{a^2 \cos^2(\alpha + \beta)}{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \sin(\alpha + \beta)}$.

A similar expression can be found for $O'C^2$ in terms of α' and β'. Then $O'C^2 = OC^2 + d^2$. This finds a and then $AB = a\sqrt{2}$.

Or thus; find as above $\tan \phi = \dfrac{\tan \alpha (1 - \tan \beta)}{\tan \beta (1 - \tan \alpha)}$.

Similarly $\tan BCO' = \dfrac{\tan \alpha' (1 - \tan \beta')}{\tan \beta' (1 - \tan \alpha')}$.

Then $OCO' = \dfrac{\pi}{2} - \phi - \phi'$ and $OC = d \cot OCO' = d \tan(\phi + \phi')$.

Thus OC is found; and then a can be determined.

31. 104·93 feet. 32. 63° 26′ 6″. 33. 30° 40′ 37″.
34. 296·40031 feet.

XVI. pages 192—199. 1. 216. 2. $\dfrac{25(\sqrt{3} - 1)}{\sqrt{3}}$.

3. 6. 8. 7 to 3; 120° 25. $\sqrt{(pq)}$; where p and q are the given segments. 46. 82° 24′ 39″; 22° 24′ 39″; 75° 10′ 42″. 50. Conversely, if this relation holds it may be shewn that the lines meet in a point.

51, 52, 53, 54 follow from the converse of 50.

XVII. pages 203—205. 1. -1, $\sqrt{3} - 4$, $-\sqrt{3} - 4$.

4. -2, $2\sqrt{2} \cos 9°$, $2\sqrt{2} \cos 63°$, $2\sqrt{2} \cos 81°$, $2\sqrt{2} \cos 153°$.

7. Let x be the height of the balloon, and a, b, c the sides of the triangle ABC; then $4c^2x^4 - 36a^2b^2x^2 + 9a^4b^4c^2 = 0$.

9. Less than 2 inches. 12. Suppose h the height of the tower, r the radius, x the distance of the first place of observation from the centre; then $\dfrac{x}{r} = \operatorname{cosec} \dfrac{\beta}{2}$, $\dfrac{x - a}{r} = \operatorname{cosec} \dfrac{\beta'}{2}$, $h = x \tan \alpha$, $h = (x - a) \tan \alpha'$. From these four equations we may eliminate x, and find h and r, and also the required relation between α, α', β, β'. 13. From the pre-

ceding question $\frac{a}{r} = \operatorname{cosec}\frac{\beta}{2} - \operatorname{cosec}\frac{\beta'}{2}$. If we suppose that an error δ of the *same* sign is made in β and β' these errors tend to compensate each other; the greatest possible error in r will be determined by supposing that errors of *opposite* signs are made in β and β'. Suppose then that instead of β we ought to have $\beta - \delta$, and instead of β' we ought to have $\beta' + \delta$. Then by Art. 194 we shall find

$$\frac{a\rho}{r^2} = \frac{\delta}{2}\left(\frac{\cos\frac{\beta}{2}}{\sin^2\frac{\beta}{2}} + \frac{\cos\frac{\beta'}{2}}{\sin^2\frac{\beta'}{2}}\right) = \frac{\delta}{2}\,\frac{\left(\cos\frac{\beta}{2} + \cos\frac{\beta'}{2}\right)\left(1 - \cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}}.$$

Divide by the value of $\frac{a}{r}$ and the required result is obtained.

14. If $PQ = a$ and $QR = b$, it may be shewn that

$$\frac{1}{SQ^2} = \frac{(a-b)^2}{4a^2b^2} + \frac{(a+b)^2}{4a^2b^2}\tan^2\beta;$$

then the change in SQ arising from a small change in β can be calculated.

XVIII. pages 207—210. 2. 1. 7. $\frac{1}{2057}$. 18. $x^2 = \frac{2}{17}(5 - 2\sqrt{2})$.

19. $x = \frac{a+b}{1-ab}$. 20. $x = 0$ or $\pm\frac{1}{2}$. 21. $x = 0$ or $\pm\frac{1}{2}$.

22. $x = -\frac{461}{9}$. 23. $x = \pm 1$ or $\pm(1 \pm \sqrt{2})$.

24. $x = a$ or $a^2 - a + 1$. 28. $x = 2$. 30. $x = 1,\ y = 2$; $x = 2,\ y = 7$.

34. $n\pi + (-1)^{m+n}\frac{\pi}{6}$, or $(m+n)\pi + (-1)^n\frac{\pi}{6}$.

35. $(2n + m)\pi \pm \frac{\pi}{3}$.

XXII. pages 246—249.

1. Use $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$. 2. Use $\sin^3 a = \frac{1}{4}(3\sin a - \sin 3a)$.

5. $\frac{n}{2}\cos a + \frac{\cos(2\theta + na)\sin na}{2\sin a}$. 9. $\frac{1}{2}(\cos 2\theta - \cos 2^{n+1}\theta)$.

10. $e^{\cos^2\theta}\cos(\theta+\cos\theta\sin\theta)$.   12. $\frac{1}{2}(e^{\sin\theta}+e^{-\sin\theta})\cos(\cos\theta)$.

13. $\dfrac{\cos\theta}{1-\cos\theta}-\log(1-\cos\theta)$.   14. $e^{\cos_2\theta}\sin\left(\dfrac{\sin 2\theta}{2}\right)$.

15. $\sin\theta-\cos\theta+e^{\sin\theta\cos\theta}\cos(\theta+\sin^2\theta)$.

20. $2^{n-2}\sin\dfrac{\theta}{2^{n-1}}-\dfrac{1}{4}\sin 2\theta$.   21. $\tan\theta-\tan\dfrac{\theta}{2^n}$.

22. $\dfrac{1}{2\sin^2\frac{1}{2}\theta}-\dfrac{2^{n-1}}{\sin^2 2^{n-1}\theta}$.   23. $\operatorname{cosec}\theta\{\cot\theta-\cot(n+1)\theta\}$.

24. $\operatorname{cosec}\left(\theta+\dfrac{\pi}{2}\right)\left\{\tan(n+1)\left(\theta+\dfrac{\pi}{2}\right)-\tan\left(\theta+\dfrac{\pi}{2}\right)\right\}$.

25. $\dfrac{\pi}{4}-\tan^{-1}\dfrac{1}{n+1}$.   26. $\tan^{-1}nx$.   27. $\dfrac{1}{2}\left(\cos\dfrac{a}{2^{n-2}}-\cos 4a\right)$.

28. $\frac{1}{2}\operatorname{cosec}\theta\{\tan(n+1)\theta-\tan\theta\}$.

29. $\dfrac{1}{4}\operatorname{cosec}\dfrac{\theta}{2}\left\{\sec\dfrac{2n+1)}{2}\theta-\sec\dfrac{\theta}{2}\right\}$.   30. $\dfrac{1}{4}\left\{\cot\dfrac{\theta}{2}-3^n\cot\dfrac{3^n\theta}{2}\right\}$.

31. $\cot^{-1}\dfrac{a}{2}-\cot^{-1}\dfrac{n+1}{2}a$.   32. $\cos\theta-\sin\theta\cot 2^n\theta$.

33. $\log 2\sin 2\theta-\dfrac{\log 2\sin 2^{n+1}\theta}{2^n}$.

XXIII. pages 265—267. 1. When $n=2$ the sum of the first series is $\dfrac{\pi^2}{6}$, and the sum of the second series is $\dfrac{\pi^2}{8}$. When $n=4$ the sum of the first series is $\dfrac{\pi^4}{90}$, and the sum of the second series is $\dfrac{\pi^4}{96}$. These results are obtained by expanding the values of og $\dfrac{\sin\theta}{\theta}$ and $\log\cos\theta$, which are given in Arts. 274 and 320, in powers of θ, and equating the coefficients of like powers.

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